(U, N)-Implications and their Characterizations

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Abstract

In this work we characterize (U, N)-implications obtained from disjunctive uninorms and continuous negations.

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1 Introduction

(U, N)-implications are some generalizations of (S, N)-implications, where a t-conorm S is replaced by a uninorm U. A similar generalization of residual implications from the setting of t-norms to the setting of uninorms has been done by De Baets and Fodor in [3]. Ruiz and Torrens have investigated quite extensively on fuzzy implications from uninorms [11] and their distributivity [10], [12].

Despite this interest, fuzzy implications obtained from uninorms are yet to be characterized. Recently, some characterizations of (S, N)-implications were given by the authors in [2]. In this work, along similar lines, we investigate and characterize (U, N)-implications obtained from continuous negations N.

After introducing the necessary preliminaries on the basic fuzzy logic operations, we list out some of the most desirable - but relevant to this work - properties of fuzzy implications and investigate their interdependencies. Following this we discuss the class of (U, N)-operations and the properties they satisfy. Finally, based on the above analysis, we derive a characterization for (U, N)-implications generated from continuous negations.

2 Basic Fuzzy Logic Operations

To make this work self-contained, we briefly mention some of the concepts and results employed in the rest of the work.

Definition 1 (see [4, 7])**.** A decreasing function $N\colon [0, 1] \to [0, 1]$ is called a fuzzy negation if $N(0) = 1$ and $N(1) = 0$. A fuzzy negation N is called

- strict if it is both strictly decreasing and continuous;
- strong if it is an involution, i.e., $N(N(x)) = x$ for all $x \in [0, 1]$.

It is well-known that if $[a, b]$ and $[c, d]$ are two closed subintervals of $[-\infty, +\infty]$ and $f\colon [a, b] \to [c, d]$ is a monotone function, then the set of discontinuous points of f is a countable subset of $[a, b]$ (see [9]). In this case we will use the pseudo-inverse $f^{(-1)}\colon [c, d] \to [a, b]$ of a decreasing and non-constant function f defined by (see [7, Sect. 3.1])

$$f^{(-1)}(y) = \sup\{x \in [a, b] \mid f(x) > y\}, \qquad y \in [c, d].$$

Lemma 1 ([2], Proposition 28)**.** *If N is a continuous fuzzy negation, then the function $\mathfrak{N}\colon [0, 1] \to [0, 1]$ defined by*

$$\mathfrak{N}(x) = \begin{cases} N^{(-1)}(x), & \text{if } x \in (0, 1], \\ 1, & \text{if } x = 0, \end{cases} \tag{1}$$

is a strictly decreasing fuzzy negation. Moreover

$$\mathfrak{N}^{(-1)} = N, \tag{2}$$

$$N \circ \mathfrak{N} = \mathrm{id}_{[0,1]}, \tag{3}$$

$$\mathfrak{N} \circ N\mid_{\mathrm{Ran}(\mathfrak{N})} = \mathrm{id}_{\mathrm{Ran}(\mathfrak{N})}. \tag{4}$$

Definition 2 (see [5])**.** An associative, commutative, increasing operation $U\colon [0, 1]^2 \to [0, 1]$ is called a uninorm, if there exists an $e \in [0, 1]$ (called the neutral element) such that

$$U(e, x) = x, \qquad x \in [0, 1].$$

Remark 1. (i) If $e = 0$, then U is a t-conorm and if $e = 1$, then U is a t-norm.

(ii) It can be easily showed, that the neutral element e corresponding to a uninorm U is unique.

(iii) For any uninorm U we have $U(0,1) \in \{0,1\}$.

(iv) A uninorm U such that $U(0,1) = 0$ is called a conjunctive uninorm and if $U(0,1) = 1$ it is called a disjunctive uninorm.

Examples of fuzzy negations, uninorms as well as the different classes of uninorms (the classes U_{min}, U_{max}, representable uninorms, idempotent uninorms) can be found in recent literature (see [4, Chap. 1], [7, Sect. 10.2], [3, 5]).

3 Fuzzy Implications

3.1 Definition and Properties

In this work the following equivalent definition proposed by Fodor and Roubens [4] is used.

Definition 3. A function $I\colon [0,1]^2 \to [0,1]$ is called a fuzzy implication operation, or a fuzzy implication, if it satisfies the following conditions:

$$I \text{ is decreasing in the first variable,} \tag{I1}$$
$$I \text{ is increasing in the second variable,} \tag{I2}$$
$$I(0,0) = 1, \tag{I3}$$
$$I(1,1) = 1, \tag{I4}$$
$$I(1,0) = 0. \tag{I5}$$

The set of all fuzzy implications will be denoted by $\mathcal{FI}$.

Directly from the above definition we see that each fuzzy implication I satisfies the following left and right boundary condition, respectively:

$$I(0,y) = 1, \qquad y \in [0,1], \tag{LB}$$
$$I(x,1) = 1, \qquad x \in [0,1]. \tag{RB}$$

Therefore, I satisfies also the normality condition

$$I(0,1) = 1. \tag{NC}$$

Consequently, every fuzzy implication restricted to the set $\{0,1\}^2$ coincides with the classical implication.

Definition 4. Let $I\colon [0,1]^2 \to [0,1]$ be any function and $\alpha \in [0,1)$. The function N_I^α given by

$$N_I^\alpha(x) = I(x,\alpha), \qquad x \in [0,1]$$

is called the natural negation of I with respect to α.

Lemma 2. *Let $I\colon [0,1]^2 \to [0,1]$ be any function and $\alpha \in [0,1)$ be arbitrary but fixed. Then the following statements are equivalent:*

(i) N_I^α is a fuzzy negation.

(ii) $I(0,\alpha) = 1$ and $I(1,\alpha) = 0$.

Proof. (i) $\Longrightarrow$ (ii) Since N_I^α is a fuzzy negation, $I(0,\alpha) = N_I^\alpha(0) = 1$ and $I(1,\alpha) = N_I^\alpha(1) = 0$.

(ii) $\Longrightarrow$ (i) This implication is obvious from the definition of a fuzzy negation.

□

It should be noted that for any $I \in \mathcal{FI}$ we have (I5), so for $\alpha = 0$ we have the natural negation $N_I = N_I^0$ of I. Also α should be less than 1, since $I(1,1) = 1$.

In the following we list out some of the desirable properties of fuzzy implications:

Definition 5. Let I be a fuzzy implication and N a fuzzy negation.

(i) I is said to have the exchange principle, if

$$I(x, I(y,z)) = I(y, I(x,z)), \tag{EP}$$

for all $x, y, z \in [0,1]$,

(ii) I is said to satisfy the law of left contraposition with respect to N if, for any $x, y \in [0,1]$,

$$I(N(x), y) = I(N(y), x). \tag{L-CP}$$

(iii) I is said to satisfy the law of right contraposition with respect to N, if, for any $x, y \in [0,1]$,

$$I(x, N(y)) = I(y, N(x)). \tag{R-CP}$$

(iv) I is said to satisfy the law of contraposition with respect to N, if, for any $x, y \in [0,1]$,

$$I(x,y) = I(N(y), N(x)). \tag{CP}$$

Lemma 3 ([2], Lemma 17)**.** *Let $I\colon [0,1]^2 \to [0,1]$ be any function and N a continuous fuzzy negation.*

(i) If I satisfies (I1) *and R-CP(N), then I satisfies* (I2).

(ii) If I satisfies (I2) *and R-CP(N), then I satisfies* (I1).

Lemma 4. *Let $I\colon [0,1]^2 \to [0,1]$ and N_I^α be a fuzzy negation for an arbitrary but fixed $\alpha \in [0,1)$.*

(i) If I satisfies (I2)*, then I satisfies* (I5).

(ii) Let I have (I2) *and* (EP)*. Then I satisfies* (I3) *if and only if I satisfies* (I4).

(iii) If I satisfies (EP)*, then I satisfies R-CP(N_I^α),*

Proof. (i) Since N_I^α is a fuzzy negation and I satisfies (I2) we get $I(1,0) \le I(1,\alpha) = N_I^\alpha(1) = 0$.

(ii) Let I have (I2) and (EP). If I satisfies (I4), then since $N_I^\alpha(0) = 1$ we have $1 = I(1,1) = I(1, N_I^\alpha(0)) = I(1, I(0,\alpha)) = I(0, I(1,\alpha)) = I(0, N_I^\alpha(1)) = I(0,0) = 1$, i.e., I satisfies (I3). The reverse implication can be shown similarly.

(iii) Since I satisfies (EP), we have $I(x, N_I^\alpha(y)) = I(x, I(y,\alpha)) = I(y, I(x,\alpha)) = I(y, N_I^\alpha(x))$, i.e., I has R-CP(N_I^α).

□

Lemma 5. *Let I be any fuzzy implication and N_I^α be a continuous fuzzy negation for an arbitrary but fixed $\alpha \in [0,1)$. If N is a strictly decreasing fuzzy negation such that $N_I^\alpha \circ N = \mathrm{id}_{[0,1]}$ and I satisfies* (EP)*, then I satisfies L-CP(N).*

Proof. By our assumptions we get

$$\begin{aligned} I(N(x),y) &= I(N(x), N_I^\alpha \circ N(y)) \\ &= I(N(x), I(N(y),\alpha)) \\ &= I(N(y), I(N(x),\alpha)) \\ &= I(N(y), N_I^\alpha \circ N(x)) \\ &= I(N(y),x), \end{aligned}$$

for any $x, y \in [0,1]$ □

Remark 2. Under the assumptions of Lemma 5, we have:

(i) If N_I^α is a strict negation, then I satisfies L-CP($(N_I^\alpha)^{-1}$).

(ii) If N_I^α is a strong negation, then I satisfies CP(N_I^α).

3.2 (S,N)-Implications and their Characterization

In this section, we give a brief introduction to one of the families of fuzzy implications that is very well studied in the fuzzy literature.

Definition 6 (cf. [1, 4, 13])**.** A function $I\colon [0,1]^2 \to [0,1]$ is called an (S,N)-implication, if there exist a t-conorm S and a fuzzy negation N such that

$$I(x,y) = S(N(x),y), \qquad x,y \in [0,1]. \tag{5}$$

If N is a strong negation, then I is called a strong implication (S-implication, for short).

The following characterization of some subclasses of (S,N)-implications is from [2], which is an extension of a result in [13].

Theorem 1 ([2])**.** *For a function $I\colon [0,1]^2 \to [0,1]$ the following statements are equivalent:*

(i) I is an (S,N)-implication generated from some t-conorm S and some continuous (strict, strong) fuzzy negation N.

(ii) I satisfies (I1)*,* (EP) *and the function N_I is a continuous (strict, strong) fuzzy negation.*

Moreover, the representation of (S,N)-implication is unique in this case.

In Theorem 1, the property (I1) can be substituted by (I2). Moreover, axioms in the above theorem are independent from each other.

4 (U,N)-Operations and (U,N)-Implications

A natural generalization of (S,N)-implications in the uninorm framework is to consider a uninorm in the place of a t-conorm.

4.1 Definition and Properties

Definition 7. A function $I\colon [0,1]^2 \to [0,1]$ is called a (U,N)-operation, if there exist a uninorm U and a fuzzy negation N such that

$$I_{U,N}(x,y) = U(N(x),y), \qquad x,y \in [0,1]. \tag{6}$$

If a (U,N)-operation is generated from U and N, then we will often denote this by $I_{U,N}$.

Proposition 1. *If $I_{U,N}$ is a (U,N)-operation, then*

(i) $I_{U,N}$ satisfies (I1)*,* (I2)*,* (I5)*,* (NC) *and* (EP)*,*

(ii) $N^e_{I_{U,N}} = N$ and $I_{U,N}$ satisfies R-CP(N),

(iii) if N is strict, then $I_{U,N}$ satisfies L-CP(N^{-1}),

(iv) if N is strong, then $I_{U,N}$ satisfies CP(N).

Proof. (i) By the monotonicity of U and N we get that $I_{U,N}$ satisfies (I1) and (I2). Moreover, it can be easily verified that $I_{U,N}$ satisfies (I5) and (NC). Finally, from the associativity and the commutativity of U we have also (EP).

(ii) For any $x \in [0,1]$ we have

$$N^e_{I_{U,N}}(x) = I_{U,N}(x,e) = U(N(x),e) = N(x).$$

Next, since $I_{U,N}$ satisfies (EP), from Lemma 4(iii) with $\alpha = e$ we have that $I_{U,N}$ satisfies R-CP(N).

(iii) If N is a strict negation, then because of Remark 2(i) we can deduce, that $I_{U,N}$ satisfies L-CP(N^{-1}).

(iv) If N is a strong negation, then because of Remark 2(ii) we can deduce, that $I_{U,N}$ satisfies CP(N).

□

If $e = 0$, then U is a t-conorm and $I_{U,N}$, as an (S, N)-implications, is always a fuzzy implication. If $e = 1$, then U is a t-norm and $I_{U,N}$ is not a fuzzy implication, since (I3) is violated. If $e \in (0, 1)$, then not for every uninorm U the (U, N)-operation is a fuzzy implication. Next results characterize these (U, N)-operation, which satisfy (I3) and (I4).

Theorem 2 (cf. [3])**.** *Let U be a uninorm with the neutral element $e \in (0, 1)$. Then the following statements are equivalent:*

(i) The function $I_{U,N}$ as defined in (6) *is a fuzzy implication.*

(ii) U is a disjunctive uninorm, i.e., $U(0, 1) = 1$.

Proof. Let U be a uninorm with the neutral element $e \in (0, 1)$.
(i) $\Longrightarrow$ *(ii)* If $I_{U,N}$ as defined in (6) is a fuzzy implication, then from (I3) we have $U(0, 1) = U(1, 0) = I_{U,N}(0, 0) = 1$.
(ii) $\Longrightarrow$ *(i)* Assume that $U(0, 1) = 1$. From Proposition 1 it is enough to show only (I3) and (I4):

$$I_{U,N}(0, 0) = U(N(0), 0) = U(1, 0) = U(0, 1) = 1,$$
$$I_{U,N}(1, 1) = U(N(1), 1) = U(0, 1) = 1.$$

□

Following the terminology used by Mas *et al.* [8] for QL-implications, only if the (U, N)-operation $I_{U,N}$ is a fuzzy implication we use the term (U, N)-implication.

Lemma 6. *Let $I_{U,N}$ be a (U, N)-implication obtained from a uninorm U with $e \in (0, 1)$ as its neutral element and continuous negation N. Let $\alpha \in (0, 1)$ be an arbitrary but fixed number. Then the following statements are equivalent:*

(i) $N^{\alpha}_{I_{U,N}} = N$.

(ii) $\alpha = e$.

Proof. Let $e \in (0, 1)$ be the neutral element of U and $\alpha \in (0, 1)$ be an arbitrary but fixed number.

(i) $\Longrightarrow$ *(ii)* If $N^{\alpha}_{I_{U,N}} = N$, then since N is continuous there exists an e' such that $e = N(e')$ and $N^{\alpha}_{I_{U,N}}(e') = I_{U,N}(e', \alpha) = U(N(e'), \alpha) = N(e') = e$. But $U(N(e'), \alpha) = U(e, \alpha) = \alpha$, because e is the neutral element of U. Hence $\alpha = e$.

(ii) $\Longrightarrow$ *(i)* On the other hand, if $\alpha = e$, then

$$N^{\alpha}_{I_{U,N}}(x) = I_{U,N}(x, \alpha) = I_{U,N}(x, e) = U(N(x), e) = N(x)$$

for all $x \in [0, 1]$, i.e., $N^{\alpha}_{I_{U,N}} = N$.

□

4.2 Characterizations of (U, N)-Implications

We start our presentation with following result.

Proposition 2. *Let I be a fuzzy implication and N any fuzzy negation. If we define a binary operation $U_{I,N}$ on $[0, 1]$ as follows*

$$U_{I,N}(x, y) = I(N(x), y), \qquad x, y \in [0, 1], \qquad (7)$$

then

(i) $U_{I,N}(x, 1) = U_{I,N}(1, x) = 1$ for all $x \in [0, 1]$, in particular $U_{I,N}(0, 1) = 1$,

(ii) $U_{I,N}$ is increasing in both the variables,

(iii) $U_{I,N}$ is commutative if and only if I has L-CP(N).

In addition, if I has L-CP(N), then

(iv) $U_{I,N}$ is associative if and only if I satisfies the exchange property (EP)*.*

(v) an arbitrary $\alpha \in (0, 1)$ is the neutral element of $U_{I,N}$ if and only if $N^{\alpha}_{I} \circ N = \mathrm{id}_{[0,1]}$.

Proof. (i) Let $x \in [0, 1]$. By the boundary condition (RB) on I we have $U_{I,N}(x, 1) = I(N(x), 1) = 1$. Also, $U_{I,N}(1, x) = I(N(1), x) = I(0, x) = 1$ again by (LB) of I.

(ii) That $U_{I,N}$ is increasing in both variables is a direct consequence of the monotonicity of I and N.

(iii) If $U_{I,N}$ is commutative, then for all $x, y \in [0, 1]$ we get $I(N(x), y) = U_{I,N}(x, y) = U_{I,N}(y, x) = I(N(y), x)$, i.e., I satisfies L-CP(N). The reverse implication can be obtained by retracing the above steps.

(iv) Let $x, y, z \in [0,1]$. If I satisfies (EP), then

$$\begin{aligned} U_{I,N}(x, U_{I,N}(y,z)) &= I(N(x), I(N(y), z)) \\ &= I(N(x), I(N(z), y)) \\ &= I(N(z), I(N(x), y)) \\ &= I(N(I(N(x), y)), z) \\ &= I(N(U_{I,N}(x,y)), z) \\ &= U_I(U_{I,N}(x,y), z). \end{aligned}$$

On the other hand, if $U_{I,N}$ is associative, then

$$\begin{aligned} I(x, I(y,z)) &= U_{I,N}(N(x), U_{I,N}(N(y), z)) \\ &= U_{I,N}(U_{I,N}(N(x), N(y)), z)) \\ &= U_{I,N}(U_{I,N}(N(y), N(x)), z)) \\ &= U_{I,N}(N(y), U_{I,N}(N(x), z)) \\ &= I(y, I(x,z)). \end{aligned}$$

(v) Let $\alpha \in (0,1)$ be arbitrary fixed. If α is the neutral element of $U_{I,N}$, then, for any $x \in [0,1]$, we have $x = U_{I,N}(x,\alpha) = I(N(x),\alpha) = N_I^\alpha(N(x))$. Conversely, if $N_I^\alpha \circ N = \mathrm{id}_{[0,1]}$, then, for any $x \in [0,1]$ we get $U_{I,N}(\alpha, x) = U_{I,N}(x,\alpha) = I(N(x),\alpha) = N_I^\alpha(N(x)) = x$ and α is the neutral element of $U_{I,N}$.

□

If N_I^α is a continuous fuzzy negation for an arbitrary but fixed $\alpha \in (0,1)$, then by Lemma 1 and previous results we can consider the modified pseudo-inverse $\mathfrak{N}_I^\alpha$ given by

$$\mathfrak{N}_I^\alpha(x) = \begin{cases} \left(N_I^\alpha\right)^{(-1)}(x), & \text{if } x \in (0,1], \\ 1, & \text{if } x = 0, \end{cases} \tag{8}$$

as the potential candidate for the fuzzy negation N in (7). Hence from Lemma 5 with $N = \mathfrak{N}_I^\alpha$ we obtain the following result.

Corollary 1 (cf. [2], Corollary 29)**.** *If a fuzzy implication I satisfies* (EP) *and N_I^α, the natural negation of I with respect to an arbitrary but fixed $\alpha \in (0,1)$, is a continuous fuzzy negation, then I satisfies* (L-CP) *with $\mathfrak{N}_I^\alpha$ from* (8).

Hence, if a fuzzy implication I satisfies (EP) and N_I^α is a continuous fuzzy negation for some $\alpha \in (0,1)$, then we conclude, that the formula (7) can be considered for the modified pseudo-inverse of the natural negation of I.

Corollary 2. *If $I \in \mathcal{FI}$ satisfies* (EP) *and N_I^α is a continuous fuzzy negation with respect to an arbitrary but fixed $\alpha \in (0,1)$, then the function U_I defined by*

$$U_I(x,y) = I(\mathfrak{N}_I^\alpha(x), y), \qquad x, y \in [0,1] \tag{9}$$

is a disjunctive uninorm with neutral element α, where $\mathfrak{N}_I$ is as defined in (8).

Theorem 3. *For a function $I\colon [0,1]^2 \to [0,1]$ the following statements are equivalent:*

(i) I is an (U,N)-operation generated from some disjunctive uninorm U with neutral element $e \in (0,1)$ and some continuous fuzzy negation N.

(ii) I is an (U,N)-implication generated from some uninorm U with neutral element $e \in (0,1)$ and some continuous fuzzy negation N.

(iii) I satisfies (I1), (I3), (EP) *and the function N_I^e is a continuous negation for some $e \in (0,1)$.*

Moreover, the representation (6) *of (U,N)-implication is unique in this case.*

Proof. That (i) is equivalent to (ii) follows immediately from Theorem 2.

(ii) $\Longrightarrow$ *(iii)* Assume, that I is an (U,N)-implication based on a uninorm U with neutral element $e \in (0,1)$ and a continuous negation N. Since every (U,N)-implication is a fuzzy implication, I satisfies (I1) and (I3). Moreover, by Proposition 1 it satisfies (EP) and $N_I^e = N$. In particular N_I^e is continuous.

(iii) $\Longrightarrow$ *(ii)* Firstly see, that from Lemma 4(iii) it follows that I satisfies (R-CP) with respect to the continuous N_I^e. Next, Lemma 3(i) implies that I satisfies (I2). Once again from Lemma 4(i) and (ii) we have that I satisfies (I3), (I4) and (I5), and hence $I \in \mathcal{FI}$. Further, by virtue of Lemmas 1 and 5 the implication I satisfies L-CP($\mathfrak{N}_I^e$). Because of Corollary 2 the function U_I defined by (9) is a disjunctive uninorm with the neutral element e.
We will show that $I_{U_I, N_I^e} = I$. Fix arbitrarily $x, y \in [0,1]$. If $x \in \mathrm{Ran}(\mathfrak{N}_I^e)$, then by (4) we have

$$\begin{aligned} I_{U_I, N_I^e}(x,y) &= U_I(N_I^e(x), y) \\ &= I(\mathfrak{N}_I^e \circ N_I^e(x), y) = I(x,y). \end{aligned}$$

If $x \notin \mathrm{Ran}(\mathfrak{N}_I^e)$, then from the continuity of N_I^e there exists $x_0 \in \mathrm{Ran}(\mathfrak{N}_I^e)$ such that $N_I^e(x) = N_I^e(x_0)$. Firstly see, that $I(x,y) = I(x_0,y)$ for all $y \in [0,1]$. Indeed, let us fix arbitrarily $y \in [0,1]$. From the continuity of N_I^e there exists $y' \in [0,1]$ such that $N_I^e(y') = y$, so

$$\begin{aligned} I(x,y) &= I(x, N_I^e(y')) = I(y', N_I^e(x)) \\ &= I(y', N_I^e(x_0)) = I(x_0, N_I^e(y')) = I(x_0, y). \end{aligned}$$

From the above fact we get

$$\begin{aligned} I_{U_I, N_I^e}(x,y) &= U_I(N_I^e(x), y) \\ &= U_I(N_I^e(x_0), y) = I(x_0, y) = I(x,y), \end{aligned}$$

so I is an (U,N)-implication.
Finally, assume that there exist two continuous fuzzy

negations N_1, N_2 and two uninorms U_1, U_2 with neutral elements $e, e' \in (0,1)$, respectively, such that $I(x,y) = U_1(N_1(x), y) = U_2(N_2(x), y)$ for all $x, y \in [0,1]$. Fix arbitrarily $x_0, y_0 \in [0,1]$. Firstly observe that from Proposition 1 we get $N_1 = N_2 = N_I^e = N_I^{e'}$. By virtue of Lemma 6 we get, that $e' = e$. Now, since N_I^e is a continuous negation there exists $x_1 \in [0,1]$ such that $N_I^e(x_1) = x_0$. Thus $U_1(x_0, y_0) = U_1(N_I^e(x_1), y_0) = U_2(N_I^e(x_1), y_0) = U_2(x_0, y_0)$, i.e., $U_1 = U_2$. Hence N and U are uniquely determined. In fact $U = U_I$ defined by (9). □

In above theorem the property (I1) can be substituted by (I2) and the property (I3) can be substituted by (I4). Moreover, the above axioms are independent from each other.

Now, the following result easily follows:

Theorem 4. *For a function $I\colon [0,1]^2 \to [0,1]$ the following statements are equivalent:*

(i) I is an (U,N)-implication generated from some disjunctive uninorm U with neutral element $e \in (0,1)$ and some strict (strong) fuzzy negation N.

(ii) I satisfies (I1), (I3), (EP) *and the function N_I^e is a strict (strong) negation.*

Once again, the representations of the (U,N)-implications described above are unique and the presented axioms are independent from each other. It is interesting, that using similar methods as in this section we are able to obtain the following characterization of (U,N)-operations.

Theorem 5. *For a function $I\colon [0,1]^2 \to [0,1]$ the following statements are equivalent:*

(i) I is an (U,N)-operation generated from some uninorm U with neutral element $e \in (0,1)$ and some continuous fuzzy negation N.

(ii) I satisfies (I1), (EP) *and the function N_I^e is a continuous negation for some $e \in (0,1)$.*

Once again, in the above theorems, the property (I1) can be substituted by (I2).

5 Concluding Remarks

In this work, we characterize (U,N)-implications obtained from disjunctive uninorms U and continuous negations N. Toward this end, we have investigated some desirable algebraic properties of fuzzy implications and obtained some characterization results. It should be noted, that (U,N)-implications are closely related with e-implications investigated in [6], whose representation is still unknown.

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