A new look on discrete quasi-copulas

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Abstract

In this paper discrete quasi-copulas (defined on a square grid I_n^2 of [0,1]) are studied and it is proved that they can be represented by means of a special class of matrices with entries in [-1,1]. Special considerations are made for the case of irreducible discrete quasi-copulas (those with range I_n) defined on the finite chain I_n, showing that they can be represented through Alternating-Sign Matrices and that they generate all discrete quasi-copulas through convex sums.

Keywords: Copula, quasi-copula, discrete scale, irreducible discrete quasi-copula, ASM matrix.

1 Introduction

It is well known the importance that copulas have in Statistics as a consequence of Sklar's theorem. This theorem shows that the joint distribution function H of two random variables X and Y having marginal distribution functions F and G, respectively, can be obtained as

$$H(x,y) = C(F(x), G(y)) \qquad \text{for all} \quad x, y \in \overline{I\!R} \quad (1)$$

where $C : [0,1]^2 \to [0,1]$ is a copula uniquely determined on $\{(F(x), G(y)) \mid x, y \in \overline{I\!R}\}$, and vice versa, given two (one dimensional) distribution functions F and G and a copula C, the function H defined by (1) is a two dimensional distribution with marginals F and G (see [15] and [12]). In fact, from this theorem, the concept of subcopula becomes essential and the problem of extending subcopulas to a copula naturally arises. A special case of subcopulas are the so-called *discrete copulas* defined on a finite subset of $[0,1]^2$, usually $I_{n,m} = I_n \times I_m$ where

$$I_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}.$$

Discrete copulas are interesting because they are related through Sklar's theorem with bivariate discrete random variables (see [10]) and they apply specially in cases where discrete ordinal structures of statistical data are essential.

Discrete copulas with domain I_n^2 are studied in [7] where it is proved that they can be represented by means of bistochastic matrices. The particular case of discrete copulas on I_n that take values again in I_n was introduced in [9] where it is proved that they can be represented through permutation matrices. Moreover, this particular type of copulas are called in [7] *irreducible discrete copulas* because, as it is proved there, all discrete copulas on I_n^2 are convex sums of irreducible discrete copulas.

On the other hand, quasi-copulas were introduced in [1] as a generalization of copulas and they were characterized in operational terms in [4]. Recently, discrete quasi-copulas defined on $I_{n,m}$ are introduced in [13] where they are represented by means of $(n \times m)$ matrices that are simply their operation table. Moreover, some methods to extend a discrete quasi-copula to a quasi-copula are presented. The special case of internal discrete quasi-copulas $C : I_n^2 \to I_n$, called here *irreducible discrete quasi-copulas* similarly to the case of copulas, has been recently introduced in [8].

Copulas and quasi-copulas can also be viewed as conjunctive aggregation operators and many papers have appeared in last years dealing with them, from this point of view (see [2], [3], [5], [7], [8], [9]).

In this paper we deal with discrete quasi-copulas defined on I_n^2 presenting a new look on them following a similar study to the one given for discrete copulas in [7] and [9]. We begin with section 2 where some preliminaries are given that will be used along the paper. In section 3 we prove that discrete quasi-copulas can be represented by means of a class of matrices with entries in the interval $[-1,1]$ that we will call *Generalized Bistochastic Matrices* or *GBM* in short. We deal first with the special case of internal discrete copulas on I_n. Discrete quasi-copulas in this special case are called *irreducible discrete quasi-copulas*, similarly to the case of copulas, and it is proved that they can be represented by the so-called *Alternating-Sign Matrices* or *ASM* matrices in short (see [14]). We devote section 4 to prove that all discrete quasi-copulas are convex sums of irreducible ones.

2 Preliminaries

In this section we give only basic definitions and properties of copulas and subcopulas. A more extensive study can be found in [12] and [4] and, for the discrete case, in [7] and [9].

Definition 1 *([12]) A (two dimensional) copula C is a binary operation on $[0,1]$, i.e., $C:[0,1]\times[0,1]\longrightarrow[0,1]$ such that*

(C1) $C(x,0)=C(0,x)=0\ \forall x\in[0,1]$

(C2) $C(x,n)=C(n,x)=x\ \forall x\in[0,1]$

(C3) $C(x,y)+C(x',y')\geq C(x,y')+C(x',y)$ *for all $x,x',y,y'\in[0,1]$ with $x\leq x', y\leq y'$ (2-increasing condition)*

Although the notion of quasi-copula was introduced in a different way, it is proved in [4] the following equivalent definition.

Definition 2 *A (two dimensional) quasi-copula Q is a binary operation on $[0,1]$, i.e., $Q:[0,1]\times[0,1]\longrightarrow[0,1]$ such that*

(Q1) $Q(x,0)=Q(0,x)=0\ \forall x\in[0,1]$ *and* $Q(x,n)=Q(n,x)=x\ \forall x\in[0,1]$

(Q2) *Q is non-decreasing in each component*

(Q3) *Q satisfies the Lipschitz condition with constant 1:*

$$|\,Q(x',y')-Q(x,y)\,|\leq|x'-x|+|y'-y|$$

for all $x,x',y,y'\in[0,1]$

The following characterization of quasi-copulas was given again in [4]:

Proposition 1 *A binary operation $Q:[0,1]\times[0,1]\longrightarrow[0,1]$ is a quasi-copula if, and only if, it satisfies $(Q1)$ and the following property:*

(Q4) *$Q(x,y)+Q(x',y')\geq Q(x,y')+Q(x',y)$ for all $x,x',y,y'\in[0,1]$ such that $x\leq x', y\leq y'$ where at least one of these four elements is equal either to 0 or to 1 (2-increasing condition on the border of $[0,1]^2$)*

Clearly each copula is a quasi-copula but not vice versa. Quasi-copulas which are not copulas are called *proper quasi-copulas.*

Definition 3 *A function $C:D\times D'\rightarrow[0,1]$, where D,D' are subsets of [0,1] containing $\{0,1\}$, is called a subcopula if it satisfies the properties of a copula for all $(x,y)\in D\times D'$.*

A special case of subcopulas are the so-called *discrete copulas* defined on a finite subset of $[0,1]^2$, usually $I_{n,m}=I_n\times I_m$ where

$$I_n=\left\{0,\frac{1}{n},\frac{2}{n},\ldots,\frac{n-1}{n},1\right\}.$$

In this paper we will consider discrete copulas defined on I_n^2. Thus, in our context, we have the following definition (see, for instance, [7]).

Definition 4 *A discrete copula C on I_n is a binary operation $C:I_n\times I_n\longrightarrow[0,1]$ satisfying properties $(C1-C3)$ for all $x,y\in I_n$.*

The range of a discrete copula C on I_n always contains I_n. When the range of C is exactly I_n we deal in fact with *internal discrete copulas $C:I_n\times I_n\rightarrow I_n$*, called *irreducible discrete copulas* in [7]. In fact, this class of discrete copulas were already introduced and studied in [9], and it is proved in [7] that all discrete copulas are convex sums of irreducible ones.

3 Discrete quasi-copulas

Following the idea of discrete copulas, we want to deal in this paper with the notion of discrete quasi-copula already introduced in [13][1].

Definition 5 *A discrete quasi-copula Q on I_n is a binary operation $Q:I_n\times I_n\longrightarrow[0,1]$ such that*

(DQ1) $Q(\frac{i}{n},0)=Q(0,\frac{i}{n})=0\ \forall\frac{i}{n}\in I_n$ *and* $Q(\frac{i}{n},1)=Q(1,\frac{i}{n})=\frac{i}{n}\ \forall\frac{i}{n}\in I_n$

(DQ2) *Q is non-decreasing in each component*

(DQ3) *Q satisfies the Lipschitz condition with constant 1.*

[1]In fact the definition could be clearly given for discrete quasi-copulas on $I_n\times I_m$, but we are interested here in the most usual case $n=m$.

It is clear that any discrete copula is a discrete quasi-copula but not vice versa and, in this sense, we will call *proper discrete quasi-copulas* to discrete quasi-copulas that are not discrete copulas.

The following characterizations can be easily derived (see [13]).

Proposition 2 *Let $Q : I_n \times I_n \to [0,1]$ be any binary operator. The following statements are equivalent:*

(i) *Q is a discrete quasi-copula on I_n.*

(ii) *Q satisfies condition (DQ1) and also*

$$0 \le Q\left(\frac{i}{n}, \frac{j}{n}\right) - Q\left(\frac{i-1}{n}, \frac{j}{n}\right) \le \frac{1}{n}$$

and

$$0 \le Q\left(\frac{i}{n}, \frac{j}{n}\right) - Q\left(\frac{i}{n}, \frac{j-1}{n}\right) \le \frac{1}{n}$$

for all $1 \le i, j \le n$.

(iii) *Q satisfies condition (DQ1) and*

$$Q\left(\frac{i}{n}, \frac{j}{n}\right) + Q\left(\frac{i'}{n}, \frac{j'}{n}\right) \ge Q\left(\frac{i}{n}, \frac{j'}{n}\right) + Q\left(\frac{i'}{n}, \frac{j}{n}\right)$$

for all $0 \le i, j, i', j' \le n$ such that $i \le i', j \le j'$ where at least one of these four elements is equal either to 0 or to n.

3.1 Irreducible discrete quasi-copulas

It is clear from condition $(DQ1)$ that the range of any discrete quasi-copula on I_n contains I_n. Let us consider, like in the case of copulas, the special case of discrete quasi-copulas with range equal to I_n, that is, discrete quasi-copulas with minimal range.

Definition 6 *An irreducible discrete quasi-copula on I_n is a discrete quasi-copula Q on I_n with range I_n.*

Thus, an irreducible discrete quasi-copula on I_n is an internal binary operation $Q : I_n \times I_n \to I_n$ with properties $(DQ1) - (DQ3)$. Note that, in order to study operations defined on a finite chain like I_n, the only important thing is the number of elements of the chain, $n + 1$ (see [11]). Then, this study is usually given for the most simple of these chains, that is,

$$L_n = \{0, 1, \ldots, n\}.$$

In this sense, we will study irreducible discrete quasi-copulas on L_n, that is, $Q : L_n \times L_n \to L_n$.

Remark 1 *It is clear that, given any irreducible discrete quasi-copula Q on L_n, the operator $Q' : I_n \times I_n \to I_n$ given by*

$$Q'\left(\frac{i}{n}, \frac{j}{n}\right) = \frac{1}{n} \cdot Q(i,j)$$

is an irreducible discrete quasi-copula on I_n. And vice versa, for any irreducible quasi-copula on I_n, Q', the operator $Q : L_n \times L_n \to L_n$ defined by

$$Q(i,j) = n \cdot Q'\left(\frac{i}{n}, \frac{j}{n}\right)$$

is an irreducible discrete quasi-copula on L_n.

Thus, from now on, we will deal in this subsection with discrete quasi-copulas on L_n. As one of the main results, we will give an useful characterization of irreducible discrete quasi-copulas in terms of a special class of matrices, generalizing the representation of copulas given in [9]. In this case, we use the so-called Alternating-Sign Matrices. For more details on this class of matrices, see [14] and the web page: http://www.research.att.com//njas/sequences/

Definition 7 *An $n \times n$ Alternating-Sign Matrix (ASM matrix) is an $n \times n$ matrix $A = (a_{ij})$ such that*

1. *$a_{ij} \in \{-1, 0, 1\} \quad \forall i, j \in \{1, 2, \ldots, n\}$*
2. *The first and the last elements $a_{ij} \ne 0$ of each row and each column are 1.*
3. *All the elements $a_{ij} \ne 0$ of each row and each column have alternating signs.*

Remark 2 *In particular, the sum of each row and each column equals 1. Observe also that a permutation matrix is an ASM matrix.*

Next we give the characterization of an irreducible quasi-copula in terms of ASM matrices.

Proposition 3 *A binary operator $Q : L_n \times L_n \longrightarrow L_n$ is a discrete quasi-copula if, and only if, there exists an $n \times n$ ASM matrix $A = (a_{ij})$ such that, for all $r, s \in L_n$,*

$$Q(r,s) = \begin{cases} 0 & \text{if } r = 0 \text{ or } s = 0 \\ \sum_{\substack{i \le r \\ j \le s}} a_{ij} & \text{otherwise} \end{cases}$$

Given the quasi-copula Q, the matrix A is obtained as

$$a_{ij} = Q(i,j) + Q(i-1,j-1) - Q(i,j-1) - Q(i-1,j) \quad (2)$$

Remark 3 *Note that, following Remark 1, the corresponding irreducible discrete quasi-copula Q' on I_n will be given by*

$$Q\left(\frac{r}{n}, \frac{s}{n}\right) = \begin{cases} 0 & \text{if } r = 0 \text{ or } s = 0 \\ \frac{1}{n} \sum_{\substack{i \le r \\ j \le s}} a_{ij} & \text{otherwise} \end{cases}$$

Corollary 1 *There is a one-to-one correspondence between the set of all irreducible discrete quasi-copulas on L_n and the set of all $n \times n$ ASM matrices. This correspondence assigns to an irreducible discrete quasi-copula Q, the ASM matrix $A = (A_{ij})$ given by (2), that will be called from now on, the associated ASM matrix of Q.*

It can be proved that if the 2-increasing condition fails for an irreducible discrete quasi-copula Q, it must fail in a square given by two consecutive rows and two consecutive columns, that is, a square determined by vertices (i, j) and $(i+1, j+1)$, for some i, j such that $0 < i, j < n-1$. Moreover, the correspondence given in Corollary 1 is such that each non-fulfilment of the 2-increasing condition in a square given by vertices (i, j) and $(i+1, j+1)$ corresponds to a negative entry (-1) in position $(i+1, j+1)$ of the associated ASM matrix and vice versa.

Thus, this matrix representation is useful in finding proper quasi-copulas simply by constructing an ASM matrix with at least one negative entry.

Example 1 *Let us consider the irreducible quasi-copula Q defined on $L_4 = \{0, 1, 2, 3, 4\}$ by the following table:*

Q	*0*	*1*	*2*	*3*	*4*
0	*0*	*0*	*0*	*0*	*0*
1	*0*	*0*	**0**	**1**	*1*
2	*0*	*0*	**1**	**1**	*2*
3	*0*	*1*	*2*	*2*	*3*
4	*0*	*1*	*2*	*3*	*4*

Then the associated ASM of Q is:

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Note that the negative entry $a_{2,3} = -1$ corresponds to the non-fulfilment of the 2-increasing condition quoted in bold face in the table of Q.

On the other hand, there are more useful applications of the given representation. For instance, the open question of finding the number of irreducible quasi-copulas on L_n was posed in [8]. From the bijection stated in Corollary 1 this question can be easily answered. The number of irreducible discrete quasi-copulas on L_n is equal to the number of $n \times n$ ASM matrices, which is known as the Robbins number (see the web page: http://www.research.att.com//njas/sequences/).
That is:

Proposition 4 *The number of irreducible discrete quasi-copulas on L_n is*

$$\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!} = \prod_{k=1}^{n} \frac{(3k-2)!}{(2n-k)!}$$

As we know (see [9]), $n!$ of them are copulas, and the other are proper quasi-copulas. For example, for $n = 3$, we have 7 quasi-copulas, one of them being a proper quasi-copula, and for $n = 4$, we have 42 quasi-copulas, and 18 of them are proper quasi-copulas. All of them can be easily listed by constructing their associated ASM matrices.

Finally, another application of our representation is due to the easy manipulation of matrices. As an example of this easy manipulation we can derive the following two properties of irreducible discrete quasi-copulas.

Proposition 5 *There is one and only one $n \times n$ ASM matrix with the diagonal formed by -1 in all positions except the first and the last ones. This ASM matrix is given by:*

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

and it corresponds to the maximum proper Archimedean discrete quasi-copula on L_n, given by:

$$Q(x, y) = \begin{cases} x - 1 & \text{if } 0 < x = y < n \\ \min\{x, y\} & \text{otherwise} \end{cases}$$

Proposition 6 *There is one and only one $n \times n$ ASM matrix with the inverse diagonal formed by -1 in all positions except the first and the last ones. This ASM* *matrix is given by:*

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -1 & 1 \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\ 1 & -1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

and it corresponds to the proper commutative discrete quasi-copula Q on L_n (close to the Lukasiewicz copula) given by:

$$Q(x,y) = \begin{cases} 1 & \text{if } x+y=n, x \neq 0, n \\ \max\{x+y-n, 0\} & \text{otherwise.} \end{cases}$$

3.2 Discrete quasi-copulas

In this subsection, we give a characterization of discrete quasi-copulas $Q : I_n \times I_n \longrightarrow [0,1]$ in terms of a new kind of matrices, that we will call Generalized Bistochastic Matrices. Let us see the definition of this class of matrices:

Definition 8 *A Generalized Bistochastic Matrix (GBM matrix) is an $n \times n$ matrix $A = (a_{ij})$ such that*

1. $\forall i, j \in \{1, 2, \ldots, n\},\ a_{ij} \in [-1, 1]$
2. $\forall j = 1, \ldots, n,\ \sum_{i=1}^{n} a_{ij} = 1,$

   $\forall i = 1, \ldots, n,\ \sum_{j=1}^{n} a_{ij} = 1.$
3. $\forall j = 1, \ldots, n,\ \forall k = 1, \ldots, n,\ 0 \leq \sum_{i=1}^{k} a_{ij} \leq 1$

   $\forall i = 1, \ldots, n,\ \forall k = 1, \ldots, n,\ 0 \leq \sum_{j=1}^{k} a_{ij} \leq 1$

Remark 4 *In particular, the first and the last elements $a_{ij} \neq 0$ of each row and each column must be strictly positive. But, contrary to the case of ASM matrices, there is not alternation of signs, that is, there can be two consecutive negative (or positive) elements.*

Now, we have the representation of quasi-copulas:

Proposition 7 *A binary operator $Q : I_n \times I_n \longrightarrow [0,1]$ is a discrete quasi-copula if, and only if, there exists a GBM matrix $A = (a_{ij})$ such that, for all $r, s \in L_n$,*

$$Q\left(\frac{r}{n}, \frac{s}{n}\right) = \begin{cases} 0 & \text{if } r = 0 \text{ or } s = 0 \\ \frac{1}{n} \cdot \sum_{\substack{i \leq r \\ j \leq s}} a_{ij} & \text{otherwise} \end{cases}$$

Given the quasi-copula Q, the matrix A is obtained as

$$a_{ij} = Q\left(\frac{i}{n}, \frac{j}{n}\right) + Q\left(\frac{i-1}{n}, \frac{j-1}{n}\right) - \left(\frac{i}{n}, \frac{j-1}{n}\right) - Q\left(\frac{i-1}{n}, \frac{j}{n}\right)$$

Example 2 *Let us consider the quasi-copula Q defined on $I_4 = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$ by the following table:*

Q	0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	1
0	0	0	0	0	0
$\frac{1}{4}$	0	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{1}{4}$
$\frac{2}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{2}{4}$
$\frac{3}{4}$	0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{2}{4}$	$\frac{3}{4}$
1	0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	1

Then the associated GBM matrix is:

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{-1}{4} & \frac{-1}{4} & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Remark 5 *Again, any non-fulfilment of Q of the 2-increasing condition, for vertices of the form $(i, j), (i+1, j+1)$, corresponds to a negative entry of A and vice versa.*

Next result follows immediately:

Corollary 2 *A discrete quasi-copula Q is commutative if and only if its associated GBM matrix is symmetric.*

Similarly to the case of copulas, idempotent elements of discrete quasi-copulas play an important role in their application. Recall that a discrete quasi-copula Q on I_n is *Archimedean* if the only idempotent elements of Q are 0 and 1.

Given an $n \times n$ matrix A, we will denote by $A^{(r)}$ the submatrix formed by the first r rows and r columns of A. Then we have the following results:

Proposition 8 *An element r is an idempotent element of a discrete quasi-copula Q if and only if $A^{(r)}$ is an $r \times r$ GBM matrix.*

Proposition 9 *Let Q be a discrete quasi-copula on I_n with associated GBM matrix A. Then Q is Archimedean if and only if $A^{(r)}$ is not a GBM matrix for any $r \in \{1, \ldots, n-1\}$.*

It is proved in [8] that any irreducible discrete quasi-copula can be represented as an ordinal sum of Archimedean irreducible discrete quasi-copulas. The same proof given there applies also for general discrete quasi-copulas obtaining the following.

Proposition 10 *Let Q be a discrete quasi-copula on I_n with the following idempotent elements: $0 = x_0 < x_1 < \ldots < x_{k-1} < x_k = 1$. Then Q is an ordinal sum*

$$Q = (\langle x_{i-1}, x_i, Q_i \rangle \mid i \in \{1, \ldots, k\}),$$

i.e., $Q(x, y) =$

$$\begin{cases} x_{i-1} + Q_i(x - x_{i-1}, y - x_{i-1}) & \text{if } (x, y) \in [x_{i-1}, x_i]^2 \\ & \text{for some } 1 \le i \le k \\ \min\{x, y\} & \text{otherwise.} \end{cases}$$

where Q_i is an Archimedean quasi-copula on $L_{(x_i - x_{i-1})}$ for each $i \in \{1, \ldots, k\}$.

Remark 6 *Note that in the above proposition, consecutive idempotents $j, j+1, \ldots, j+s$ correspond to trivial ordinal summands Q_i on L_1 where the only possible quasi-copula is the minimum. Thus, in this case Q is also given by the minimum in $[j, j+s]^2$.*

Let us now introduce the concept of ordinal sum of GBM matrices closely related to the ordinal sum of discrete quasi-copulas.

Definition 9 *Let n_i be a positive integer and A_i an $n_i \times n_i$ GBM matrix for $i = 1, \ldots, k$. Let $n = n_1 + \ldots + n_k$ and define an $n \times n$ GBM matrix, A, as*

$$A = \begin{pmatrix} A_1 & 0 & 0 & \ldots & 0 \\ 0 & A_2 & 0 & \ldots & 0 \\ 0 & 0 & A_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & A_k \end{pmatrix}$$

We will call A the ordinal sum of the matrices $A_1, \ldots, A_k$ and we will denote it by $A = A_1 \oplus \ldots \oplus A_k$.

Then, we have the following correspondence between both concepts of ordinal sums.

Proposition 11 *Let Q be an ordinal sum of discrete Archimedean quasi-copulas of the form*

$$Q = (\langle x_{i-1}, x_i, Q_i \rangle \mid i \in \{1, \ldots, k\}).$$

Then, its associated GBM matrix is the ordinal sum of the GBM matrices A_i, where each A_i is the associated $(x_i - x_{i-1}) \times (x_i - x_{i-1})$ GBM matrix of Q_i for $i = 1, \ldots, k$. Moreover, the same is true for irreducible discrete quasi-copulas and their associated ASM matrices.

Note that, in the above correspondence between ordinal sums of discrete quasi-copulas and matrices, if we have a trivial ordinal summand Q_i of a quasi-copula Q, then we will also have the trivial (1×1) GBM matrix associated to this summand, in the ordinal sum of GBM matrices corresponding to Q.

4 Convex set of discrete quasi-copulas

It is proved in [7] that the set of all discrete copulas is the convex closure of all irreducible discrete copulas. The same statement is true for discrete quasi-copulas:

Proposition 12 *The class of all discrete quasi-copulas is the smallest convex set containing the class of all irreducible discrete quasi-copulas.*

A consequence of this proposition is that any discrete quasi-copula is a convex linear combination of irreducible discrete quasi-copulas. Let us give next the algorithm that, given the GBM matrix A of a discrete quasi-copula Q, produces the ASM matrices corresponding to the irreducible quasi-copulas of the convex linear combination.

Algorithm

Let $A = (a_{ij})$ be a GBM matrix.

1) Let $c_1 = \min\{|a_{ij}| : a_{ij} \neq 0\}$
2) If $c_1 = 1$, then A is an ASM matrix and the process has finished.

   If not, let us construct an ASM matrix A_1 with only the following restrictions: We put a 1 (if $a_{ij} > 0$) or a -1 (if $a_{ij} < 0$) in position (i, j) and 0's in all positions where A has 0's. If the minimum c_1 is reached in more than one position, then we can put a 1 or a -1 (depending on wether $a_{ij} > 0$ or < 0) in any one of these positions.
3) Let $A_1^* = \dfrac{1}{1 - c_1}(A - c_1 \cdot A_1)$. Note that equivalently, we will have

$$A = c_1 \cdot A_1 + (1 - c_1) \cdot A_1^*.$$

A_1^* is a GBM matrix. If it is an ASM matrix, the algorithm is finished. If not, we apply steps 1, 2 and 3 to the matrix A_1^*.

This process ends up with the list of ASM matrices and the coefficients of the convex linear combination are easily obtained from the algorithm.

Example 3 *Let us consider the quasi-copula Q given in example 2. Its associated GBM matrix is:*

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{-1}{4} & \frac{-1}{4} & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The first step of the algorithm gives $c_1 = \frac{1}{4}$, reached for instance in position a_{22}. Thus, we can choose the following ASM matrix:

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Then, the matrix $A_1^* = \dfrac{1}{1-c_1}(A - c_1 \cdot A_1)$ is

$$\begin{pmatrix} \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{-1}{3} & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

A_1^* is a GBM matrix, but it is not an ASM matrix. Thus we apply steps 1, 2 and 3 to this matrix. Now we have $c_2 = \frac{1}{3}$, reached in a_{23}, and we choose an appropriate A_2:

$$A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Next, we calculate the corresponding A_2^*:

$$A_2^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

which results to be an ASM matrix. Thus the algorithm has finished. Finally, we obtain the convex combination:

$$A = \frac{1}{4}A_1 + \frac{3}{4}A_1^* = \frac{1}{4}A_1 + \frac{3}{4}\left(\frac{1}{3}A_2 + \frac{2}{3}A_2^*\right) =$$

$$\frac{1}{4}A_1 + \frac{1}{4}A_2 + \frac{1}{2}A_2^*.$$

Consequently, the discrete quasi-copula Q is given by the convex linear combination:

$$Q = \frac{1}{4}Q_1 + \frac{1}{4}Q_2 + \frac{1}{2}Q_2^*$$

where Q_1, Q_2 and Q_2^* are the irreducible discrete quasi-copulas with associated ASM matrices A_1, A_2 and A_2^*, respectively.

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