Cut equivalence of fuzzy relations

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Abstract

Fuzzy relations on the same domain are classified according to the equality of families of cut sets. This equality of fuzzy relations is completely characterized, not only for unit interval valued fuzzy relations, but more generally, for fuzzy relations whose domain is a (complete) lattice. Similar results for fuzzy sets in general are considered in [12]. Applications of the results for fuzzy congruence relations are also presented.

Keywords: equivalence of fuzzy relations, lattice valued fuzzy relations

1 Lattice valued fuzzy relations

Fuzzy relations are considered as mappings from a direct product of sets into a complete lattice L. Most of the results are deduced for the direct product of two sets, although they are valid for the product of any finite family of sets. L can be the unit interval $[0, 1]$ of real numbers, and in this case we obtain ordinary fuzzy relations. However, we consider the general case where L is any complete lattice. Such relations are called **L-fuzzy relations**, or **lattice valued fuzzy relations**. The most often considered special case of fuzzy relations is the one when two sets in the direct product are equal. In this case, we call a fuzzy relation $R : X \times X \to L$, a fuzzy relation on X.

If $R : A \times B \to L$ is a fuzzy relation on sets A and B, then for $p \in L$, the set $R_p := \{(x, y) \in A \times B \mid R(x, y) \geq p\}$ is a **p-cut**, or a **cut relation** of R. The collection of all cut relations of R is denoted by R_L, that is, $R_L := \{R_p \mid p \in L\}$.

The following is a known property of fuzzy relations.

Proposition 1 *Let $R : A \times B \to L$ be a fuzzy relation on A and B. Then the collection $R_L = \{R_p \mid p \in L\}$ of cut relations of R is a complete lattice under the set inclusion.*

In the following, an equivalence relation on L is introduced, related to cuts of a fuzzy relation $R : A \times B \to L$:

Let $\approx$ be a binary relation on L, such that for $p, q \in L$

$p \approx q$ if and only if $R_p = R_q$.

By $R(A \times B)$ we denote the set of images of a fuzzy relation:

$R(A \times B) = \{p \in L \mid p = R(x, y),$ for some $x \in A, y \in B\}$.

The following proposition gives a characterization of the relation $\approx$. As usual, we denote by $\uparrow p$ the principal filter in a lattice L, generated by $p \in L$:

$\uparrow p := \{x \in L \mid p \leq x\}$.

Proposition 2 *If R is a fuzzy relation on A and B and $p, q \in L$, then*

$p \approx q$ *if and only if* $\uparrow p \cap R(A \times B) = \uparrow q \cap R(A \times B)$.

The relation $\leq$ in the lattice L induces an order on the set of equivalence classes modulo $\approx$, i.e., on $L/\approx$, in the following way: for $p, q \in L$, let

$[p]_\approx \leq [q]_\approx$ if and only if

$$\uparrow q \cap R(A \times B) \subseteq \uparrow p \cap R(A \times B). \qquad (1)$$

It is not difficult to prove that the above relation $\leq$ is an ordering relation on $L/\approx$. This partially ordered set is anti-isomorphic with the poset of cut sets of R considered under inclusion.

Proposition 3 *If R is an L-fuzzy relation on $A\times B$, then:*

$$[p]_{\approx} \leq [q]_{\approx} \text{ if and only if } R_q \subseteq R_p.$$

Since the mapping $p \mapsto \bigvee[p]_{\approx}$ $(p \in L)$ is a closure operation on L, $L/\approx$ is a quotient in L. Therefore, we have the following proposition.

Proposition 4 *Let $R : A\times B \to L$ be an L-fuzzy relation. Then the poset $(L/\approx, \leq)$ is a lattice, anti-isomorphic with the lattice $(R_L, \subseteq)$ of cuts of R.*

2 Equivalence of fuzzy relations

Let L be a complete lattice, A and B nonempty sets, and $\mathcal{R}_L(A \times B)$ the collection of all fuzzy relations on A and B whose co-domain is L.

Let

$$L_R := (\{\uparrow p \cap R(A \times B) \mid p \in L\}, \subseteq).$$

L_R consists of particular collections of images of R in L and it is a poset under inclusion.

Proposition 5 *If $R : A \times B \to L$ is a fuzzy relation on $A \times B$, then the poset L_R is a lattice which is isomorphic with the lattice R_L of cuts of R.*

Let $\mathcal{R}_L(A \times B)$ be the set of all lattice valued fuzzy relations on sets A and B, where L is a fixed complete lattice. $\mathcal{R}_L(A \times B)$ is a lattice itself, under the natural order, induced by the one from the lattice L:

if $R, S \in L(A \times B)$, then $R \leq S$ if and only if for each $(x, y) \in A \times B$, $R(x, y) \leq S(x, y)$.

Since for an infinite lattice there are infinitely many fuzzy relations even if A and B are finite sets, it is essential to build a natural classification of elements in the set $\mathcal{R}_L(A \times B)$.

In the following we introduce an equivalence relation on the set $\mathcal{R}_L(A\times B)$, which turns out to be the foundation of the mentioned classification.

Definition

Let $\sim$ be the relation on $\mathcal{R}_L(A \times B)$, defined as follows:

$R \sim S$ if and only if the correspondence f : $R(x, y) \mapsto S(x, y)$, $(x, y) \in A \times B$ is a bijection from $R(A \times B)$ onto $S(A \times B)$ which can be extended to the isomorphism from the lattice L_R onto the lattice L_S, given by the map

$F(\uparrow p \cap R(A \times B)) := \uparrow\bigwedge\{S(x, y) \mid S(x, y) \geq p\} \cap S(A \times B)$, $p \in L$. $(*)$

Observe that we extend a correspondence from L to L to the correspondence from a family of subsets of L to a family of subsets of L. The extension means that for every $(x, y) \in A \times B$, we consider the set $\uparrow R(x, y) \cap R(A\times B)$ instead of an element $R(x, y) \in L$.

It is easy to prove that the map F is well defined.

Proposition 6 *The relation $\sim$ is an equivalence relation on $\mathcal{R}_L(A \times B)$.*

If $R \sim S$, then the fuzzy relations R and S on $A \times B$ are said to be **equivalent**.

In the paper [6] the notion of equivalence of fuzzy sets (on $[0, 1]$ real interval) is defined in the following way:

Two fuzzy sets on the same set X are equivalent if for all $x, y \in X$

$\mu(x) < \mu(y)$ if and only if $\nu(x) < \nu(y)$ and $\mu(x) = 0$ if and only if $\nu(x) = 0$.

In the mentioned paper it is proved that for fuzzy sets with the unit interval co-domain and with finite images, this condition is equivalent with the condition that μ and ν have equal families of cut sets.

However, such an equivalence does not hold for unit interval valued fuzzy sets with arbitrary images, and for lattices valued fuzzy sets it fails even in finite case. In the paper [12] a new condition for equality of fuzzy sets is introduced for fuzzy sets in general. Here we state that the condition analogous to the one from the paper [6] is a consequence of the equivalence of fuzzy relations introduced here.

Proposition 7 *Let $R, S \in \mathcal{R}_L(A \times B)$ and $R \sim$*

S. Then for all $x, z \in A$ and $y, t \in B$,

$R(x,y) \le R(z,t)$ if and only if $S(x,y) \le S(z,t)$. $(**)$

If R and T are fuzzy relations with finite number of values in the interval $[0,1]$, the above condition $(**)$ is also sufficient in order for R and T to be equivalent, which is a consequence of the results in [6]. The counter-example in case of lattice valued fuzzy sets is given in [12].

Another difference with the results in [6] is that we do not formally distinguish between the bottom (0) and any other element p of the lattice L. Therefore, we do not require (as in [6]) that the supports of equivalent fuzzy sets coincide, since 0-cut is a function like any other p-cut, $p \in L$. However, the results would be very similar if we had transformed the present conditions to the ones in which the corresponding supports for fuzzy sets (or relations) were equal.

The following theorem is a direct consequence of the theorem proved in [12] for lattice valued fuzzy sets.

Theorem 1 *Let $R, S : A \times B \to L$ be two fuzzy relations. Then $R \sim S$ if and only if fuzzy relations R and S have equal families of cut relations.*

3 Equivalent fuzzy similarity and ordering relations

The above considerations can be applied to some special classes of fuzzy relations, in particular on fuzzy similarity and fuzzy ordering relations.

We will just briefly recall notions of a lattice valued fuzzy similarity relation and a lattice valued fuzzy ordering relations.

An L-valued relation S on X (i.e., a mapping from X^2 to L) is a **similarity relation** (**fuzzy equivalence**) on X if it is

reflexive: $S(x,x) = 1$, for every $x \in X$ (1 is a top element of L);

symmetric: $S(x,y) = S(y,x)$, for all $x, y \in X$;

transitive: $S(x,y) \wedge S(y,z) \le S(x,z)$, for all $x, y, z \in X$.

If S is reflexive and transitive, then it is an **L-valued quasi-ordering** relation on X.

An L-valued fuzzy relation S on X is **fuzzy ordering relation** if it is reflexive, transitive and

anti-symmetric: $S(x,y) \wedge S(y,x) = 0$, where 0 is the bottom element of the lattice.

It is well known and easy to prove that every cut relation of a similarity relation is a crisp equivalence relation on A and also every cut relation (except the 0-cut) of a fuzzy ordering relation is a crisp ordering relation on A.

Fuzzy similarity and ordering relations are important in many applications. Every such fuzzy relation uniquely determines a family of crisp equivalences or a family of crisp ordering relations. Fuzzy relations are essentially equivalent when they possess the same cut relations. Therefore, the consequences of Theorem 1 says when two fuzzy similarity and two fuzzy ordering relations are equivalent.

Corollary 1 *Let $R, S : A^2 \to L$ be two fuzzy similarity relations. Then $R \sim S$ if and only if R and S have equal families of cut equivalence relations.*

Corollary 2 *Let $R, S : A^2 \to L$ be two fuzzy ordering relations. Then $R \sim S$ if and only if fuzzy sets R and S have equal families of cut ordering relations.*

Example. By the tables below, we present two L-valued equivalence relations R, S on the set X, where $X = \{x, y, z, t\}$, and L is the lattice in Fig. 1.

R	x	y	z	t
x	1	c	e	0
y	c	1	0	0
z	e	0	1	g
t	0	0	g	1

S	x	y	z	t
x	1	b	f	0
y	b	1	0	0
z	f	0	1	a
t	0	0	a	1

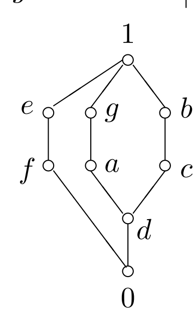
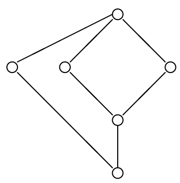


Figure 1

Cut-sets of these equivalence relations are crisp equivalences, given in the sequel by two collections of the corresponding partitions:

R_L :	S_L :
$R_1 = \{\{x\},\{y\},\{z\},\{t\}\}$	$S_1 = \{\{x\},\{y\},\{z\},\{t\}\}$
$R_g = \{\{x\},\{y\},\{z,t\}\}$	$S_g = \{\{x\},\{y\},\{z\},\{t\}\}$
$R_b = \{\{x\},\{y\},\{z\},\{t\}\}$	$S_b = \{\{x,y\},\{z\},\{t\}\}$
$R_c = \{\{x,y\},\{z\},\{t\}\}$	$S_c = \{\{x,y\},\{z\},\{t\}\}$
$R_a = \{\{x\},\{y\},\{z,t\}\}$	$S_a = \{\{x\},\{y\},\{z,t\}\}$
$R_d = \{\{x,y\},\{z,t\}\}$	$S_d = \{\{x,y\},\{z,t\}\}$
$R_e = \{\{x,z\},\{y\},\{t\}\}$	$S_e = \{\{x\},\{y\},\{z\},\{t\}\}$
$R_f = \{\{x,z\},\{y\},\{t\}\}$	$S_f = \{\{x,z\},\{y\},\{t\}\}$
$R_0 = \{\{x,y,z,t\}\}$	$S_f = \{\{x,y,z,t\}\}$



$R_L = S_L \cong L_R \cong L_S$

Figure 2

These collections of cut-sets coincide, forming a lattice under the order dual to inclusion. It is easy to check that this lattice is isomorphic with both, L_R and L_S, since $R \sim S$. All these isomorphic lattices can be represented by the diagram in Fig. 2.

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