A first approximation to N_g- Coherence Measures *

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Abstract

Focusing exclusively on the semantics of the words *coherence* and *contradiction*, it looks like they could be related concepts, considering that the more contradictory a couple of elements are, the less coherent they appear to be, and viceversa. Unfortunately, reviewing the concepts of contradiction degrees studied in [1] and [2], and coherence measures examined at length in [5], it is clear that, with these measures, equivalence does not hold. This paper deals with the relation between the two concepts, and the possible generalization of the coherence measure to find a wider range of sets where the desired equivalence holds.

Keywords: Coherence Measures, Contradiction Measures

1 Introduction. Coherence and contradiction in finite sets

Many tools can be used in to compare fuzzy sets. Two of such tools defined and used recently are coherence and contradiction measures. Given a finite set $X=\{x_1,\ldots,x_n\}$ and the set of fuzzy sets over X, $\mathcal{F}(X)$, a coherence measure is defined as a measure $cohe : \mathcal{F}(X)\times\mathcal{F}(X)\to[0,1]$ having the following properties [5].

i) $cohe(\mu,\sigma) = cohe(\sigma,\mu)$

ii) $cohe(\mu,\sigma^c) = 1 - cohe(\mu,\sigma)$

iii) $cohe(\emptyset,X) = 0$

for all μ,σ fuzzy sets in $\mathcal{F}(X)$, where σ^c is the usual complement given by $\sigma^c(x) = 1-\sigma(x)$ for all x.

As we shall see, this definition is too general, and a coherence measure can be determined by just setting its values on a subset of $\mathcal{F}(X)$. This generality may be caused by the lack of monotonicity derived from the definition of the measure itself. Looking for a more restrictive, ultimately monotonic, definition for a coherence measure, we may wonder if a contradiction measure ([1]) could be the negation of a coherence measure. If so, we would have found a more restrictive definition, since a contradiction measure has the property of anti-monotonicity. Unfortunately, however, the equivalence may hold for only some and not all pairs of sets in $\mathcal{F}(X)$. With the aim of finding a bigger set of fuzzy sets where a coherence measure could be the negation of a contradiction measure, we will generalize *cohe* using not only the usual negation $N(x) = 1-x$, but any other strong negation $Ng(x) = g^{-1}(1-g(x))$, where g is an order automorphism. With this generalization and using a fixed Ng, we will also extend some of the work presented in [5].

From now on we shall consider X to be a finite set $X = \{x_1,\ldots,x_n\}$, and μ_k the constant fuzzy set $\mu_k(x) = k$ for all x in the universe.

2 Generality of a coherence measure

To study the generality of a coherence measure, let us firstly review some properties of *cohe* trivially

* This work has been partially supported by the Spanish CICYT under projects PB98-1379-C02-02 (MEC) and TIC00-1420
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deduced from its definition.

Lemma 2.1([5])

Let *cohe*: $\mathcal{F}(X) \times \mathcal{F}(X) \rightarrow [0,1]$ be a coherence measure and μ, σ fuzzy sets in $F(X)$. It is trivial to prove:

$i)$ $cohe(\mu^c, \sigma^c) = cohe(\mu, \sigma)$

$ii)$ $cohe(\emptyset, \emptyset) = cohe(X, X) = 1$

$iii)$ For all $\mu \in \mathcal{F}(X)$, $cohe(\mu, \mu_{0.5}) = 0.5$

Now, to determine a coherence measure, it will be useful to define the following subsets of $\mathcal{F}(X)$:

$M_1 = \{\mu \in F(X) / \mu(x_{min\{i/\mu(x_i) \neq 0.5\}}) < 0.5\}$
$M_2 = \{\mu \in F(X) / \mu(x_{min\{i/\mu(x_i) \neq 0.5\}}) > 0.5\}$

Obviously, $\mathcal{F}(X) = M_1 \cup M_2 \cup \mu_{0.5}$.

Now, to define a coherence measure *cohe* , it is enough to set $cohe(\mu, \mu_{1/2})$ = 0.5 for any μ, $cohe(\emptyset, X) = 0$, and to choose the values of *cohe* for pairs of sets in M_1. In fact, the other values are obtained by applying the two first properties of a coherence measure as follows:

$i)$ if $\mu \in M_1, \sigma \in M_2$, then $cohe(\mu, \sigma) = 1 - cohe(\mu, \sigma^c)$, defined above, since $\sigma^c \in M_1$

$ii)$ if $\mu \in M_2, \sigma \in M_1$, then $cohe(\mu, \sigma) = cohe(\sigma, \mu)$ defined by $i)$

$iii)$ if $\mu \in M_2, \sigma \in M_2$, then $cohe(\mu, \sigma) = 1 - cohe(\mu, \sigma^c) = 1 - cohe(\sigma^c, \mu) = cohe(\sigma^c, \mu^c)$ defined above, since $\mu^c, \sigma^c \in M_1$

Note, firstly, that no monotonicity condition is demanded. So, for example, if a coherence measure *cohe* is determined by:

$cohe(\emptyset, X) = 0$, $cohe(\mu, \mu_{0.5}) = 0.5 \;\; \forall \mu \in F(X)$, and $cohe(\mu, \sigma) = 0 \; \forall \mu, \sigma \in M_1$, this will lead us to

$cohe(\mu_{0.499}, \mu_{0.499}) = 0$ and
$cohe(\mu_{0.499}, \mu_{0.5}) = 0.5$

$cohe(\mu_{0.499}, \mu_{0.501}) = 1$ and
$cohe(\mu_{0.501}, \mu_{0.5}) = 0.5$

The example shows that the coherence values for three sets no matter how close, are very different, when they should be very similar considering only the semantics of the term *coherence*

To restrict a coherence measure, we may be tempted to see if a coherence measure could be the negation of a contradiction measure, since the semantics of both terms seems to be very related in this respect. The following definition was established in [1].

Definition 2.2. Given the universe of discourse X, a function $Ct : \mathcal{F}(X) \times \mathcal{F}(X) \rightarrow [0,1]$ is said to be a measure of contradiction between fuzzy sets if ([1]):
i) $Ct(\mu, \mu) = 0$ for all normalised μ
ii) $Ct(\mu_\emptyset, \mu_\emptyset) = 1$
iii) $Ct(\mu, \eta) = Ct(\eta, \mu)$
iv) If $\mu \leq \sigma$, then $Ct(\mu, \eta) \geq Ct(\sigma, \eta)$, for all η.

An example of a contradiction measure is $Ct(\mu, \eta) = Max(0, 1 - Sup_x(\mu(x) + \eta(x)))$.

Lemma 2.3 A coherence measure cannot be a contradiction measure.

Proof. In fact, if $Co : \mathcal{F}(X) \times \mathcal{F}(X) \rightarrow [0,1]$ is a coherence measure, $Co(X, X) = 1$, and if it is also a contradiction, $Co(X, X) = 0$, attaining a contradiction.

Lemma 2.4 A coherence measure cannot be the negation of a contradiction measure.

Proof. If $Co : \mathcal{F}(X) \times \mathcal{F}(X) \rightarrow [0,1]$ is a coherence measure, $Co(\emptyset, \emptyset) = 1$, and if it is also the negation of a contradiction, $Co(\emptyset, \emptyset) = 1 - Ct(\emptyset, \emptyset) = 1 - 1 = 0$, attaining an absurd.

Nevertheless, given a contradiction measure, for example, the above-mentioned $Ct(\mu, \eta) = Max(0, 1 - Sup_x(\mu(x) + \eta(x)))$, it is possible to define a coherence measure that is coincidental with $1 - Ct$ on the subsets in M_1, by setting $cohe(\mu, \sigma) = 1 - Ct(\mu, \sigma)$ for all $\mu, \sigma \in M_1$, and proceeding as above.

3 Generalization of a Coherence Measure

Firstly, we should recall that strong negations were characterised in [6]: $N : [0,1] \rightarrow [0,1]$ is a strong negation if there is an order automorphism $g : [0,1] \rightarrow [0,1]$, with $g(0) = 0$, $g(1) = 1$, such that $N = g^{-1}(1 - g)$. In this case, N will be denoted as N_g. The fixed point of this negation is

$n_g = g^{-1}(1/2) \in (0,1)$.

Additionally, the usual negation function $1 - id$ is implicit in the second property of the definition of a coherence measure. The use of this negation is closely related to both property iii) from lemma 2.1. and the definition of the set $M_1 = \{\mu \in \mathcal{F}(X)/\mu(x_{min\{i/\mu(x_i)\neq 0.5\}}) < 0.5\}$ used above. We may wonder if the use of another strong negation $N_g = g^{-1}(1-g(x))$ will alter the behaviour of the measure or, at least, the generality shown by the usual definition.

Definition 3.1

Given a finite set X=$\{x_1,...,x_n\}$, the set of fuzzy sets over X, $\mathcal{F}(X)$, and a strong negation $N_g(x) = g^{-1}(1-g(x))$; a generalized coherence measure with respect to g, $cohe_g$, is defined as a measure $\mathcal{F}(X)\times\mathcal{F}(X) \to [0,1]$ having the following properties.

$i)$ $cohe_g(\mu,\sigma) = cohe_g(\mu,\sigma)$

$ii)$ $cohe_g(\mu,N_g(\sigma)) = N_g(cohe(\mu,\sigma))$

$iii)$ $cohe_g(\emptyset,X) = 0$

with μ,σ fuzzy sets in $\mathcal{F}(X)$.

Equivalent properties to the usual negation can be trivially proved:

Lemma 3.2. Let $cohe_g$:$\mathcal{F}(X)\times\mathcal{F}(X) \to [0,1]$ be a generalized coherence measure with respect to g and μ,σ fuzzy sets in $\mathcal{F}(X)$. It is trivial to prove:

$i)$ $cohe_g(N_g(\mu),N_g(\sigma)) = cohe_g(\mu,\sigma)$

$ii)$ $cohe_g(\emptyset,\emptyset) = cohe_g(X,X) = 1$

$iii)$ $\forall\mu \in F(X)$, $cohe_g(\mu,\mu_{n_g}) = n_g$

Again, we can define the subsets

$M_{g1} = \{\mu \in \mathcal{F}(X)/\mu(x_{min\{i/\mu(x_i)\neq n_g\}}) < n_g\}$
$M_{g2} = \{\mu \in \mathcal{F}(X)/\mu(x_{min\{i/\mu(x_i)\neq n_g\}}) > n_g\}$

It is clear that $F(X) = M_{g1} \cup M_{g2} \cup \mu_{n_g}$, and a N_g-coherence $cohe_g$ will be determined by its values on each pair of fuzzy sets in M_{g1}, provided $cohe_g(\emptyset,X) = 0$ and $cohe_g(\mu,\mu_{n_g}) = n_g$ for all $\mu \in F(X)$.

The proof of the following lemma is similar to that of the lemma 2.4.

Lemma 3.3. A N_g-coherence measure cannot be the negation of a N_g-contradiction measure.

Nevertheless, it should be pointed out that, given a N_g-contradiction measure, it is always possible to define a N_g-coherence measure in such a way that both are coincidental on M_{g1}. Furthermore, it should be noted that the closer n_g is to 1, the larger the subset of $\mathcal{F}(X)$ in which a N_g-coherence measure and the negation of a N_g-contradiction measure can be coincidental is.

4 Coherence measures determined by a function

Having observed the generality of coherence measures given in [5], we may want to get these measures in a more demanding way. This can be done using functions $\mathcal{F} : [0,1]^2 \to [0,1]$ on the degrees of membership of each element in the universe. Accordingly,

Lemma 4.1. Let $X = \{x_1,x_2,...,x_m\}$ be a finite set of m elements and two fuzzy sets μ and σ in $\mathcal{F}(X)$. Let g be an order isomorphism, N_g its associated strong negation, and $f : [0,1]^2 \to [0,1]$. Let us consider β_g : $\mathcal{F}(X)\times\mathcal{F}(X) \to [0,1]$ defined as follows:

$\beta_g(\mu,\sigma) = g^{-1}(\sum_{i=1}^{m} f(\mu(x_i),\sigma(x_i)))$

Then β_g is a N_g-coherence measure if and only if for all $x,y \in [0,1]$

a) $f(x,y) = f(y,x)$

b) $f(x,N_g(y)) = (1/m) - f(x,y)$

c) $f(0,1) = 0$

Finally, this result suggests that N_g-ambiguity measures can be obtained similarly.

Lemma 4.2. Let $X = \{x_1,x_2,...,x_m\}$ be a finite set of m elements, the function $h : [0,1] \to [0,1]$, and the strong negation $N_g = g^{-1}(1-g)$. Then the function $\alpha_g : \mathcal{F}(X) \to [0,1]$, given by

$$\alpha_g(\mu) = g^{-1}(\sum_{i=1}^{m} h(\mu(x_i)))$$

is a N_g-ambiguity measure if and only if:
a) $h(0)=0$
b) $h(a)=h(N_g(a))$, for all $a\in[0,1]$.

5 Ambiguity measures and coherence measures

The idea of extending an ambiguity measure to a coherence measure was studied in [5]. We may wonder if it is possible to extend the former results for a generalized ambiguity measure with the generalization of the coherence measure .

In [3] and [4], Fishburn established that a measure $\alpha : P(E)\rightarrow[0,1]$, where $P(E)$ is the set of classical subsets of E, is a measure of ambiguity if the following conditions hold:
i) $\alpha(\emptyset)=0$
ii) $\alpha(\mu^c)=\alpha(\mu)$
iii) $\alpha(\mu\cap\sigma)+\alpha(\mu\cup\sigma)\leq\alpha(\mu)+\alpha(\sigma)$

Yager extended this definition to the case of fuzzy subsets in [8], by simply demanding the same conditions, taking the definitions of complement, intersection and union given by Zadeh in [9], that is, $\mu^c(x)=1-\mu(x)$, $(\mu\cup\sigma)(x)=Max(\mu(x),\sigma(x))$, $(\mu\cap\sigma)(x)=Min(\mu(x),\sigma(x))$.

Now, given a strong negation $N_g=g^{-1}(1-g)$, a function

$\alpha_g:\mathcal{F}(X)\rightarrow[0,1]$

is a generalized ambiguity measure if and only if the following axioms hold:

i) $\alpha(\emptyset)=0$

ii) $\alpha(\mu)=\alpha(N_g(\mu))$

iii) $g(\alpha(\mu\cup\sigma))+g(\alpha(\mu\cap\sigma))\leq g(\alpha(\mu))+g(\alpha(\sigma))$

It is trivial to prove that

Lemma 5.1. Let $cohe_g:\mathcal{F}(X)\times\mathcal{F}(X)\rightarrow[0,1]$ be a N_g-coherence measure. Then the function $\alpha:\mathcal{F}(X)\rightarrow[0,1]$ defined by

$$\alpha(\mu)=N_g(cohe_g(\mu,\mu))$$

is a N_g-ambiguity measure if and only if

$g(cohe_g(\mu,\mu))+g(cohe_g(\sigma,\sigma))\leq g(cohe_g(\mu\cup\sigma,\mu\cup\sigma)+g(cohe_g(\mu\cap\sigma,\mu\cap\sigma)))$.

Proof. In fact,

i) $\alpha(\emptyset)=N_g(cohe_g(\emptyset,\emptyset))=N_g(1)=0$
ii) $\alpha(\mu)=N_g(cohe_g(\mu,\mu))=N_g(cohe_g(N_g(\mu),N_g(\mu)))=\alpha(N_g(\mu))$
iii) As $g(N_g(x))=g(g^{-1}(1-g(x)))=1-g(x)$,

$g(\alpha(\mu\cup\sigma))+g(\alpha(\mu\cap\sigma))\leq g(\alpha(\mu))+g(\alpha(\sigma))$

iff

$g(N_g(cohe_g(\mu\cup\sigma,\mu\cup\sigma)))+g(N_g(cohe_g(\mu\cap\sigma,\mu\cap\sigma)))\leq g(N_g(cohe_g(\mu,\mu)))+g(cohe_g(\sigma,\sigma)))$

iff

$1-g(cohe_g(\mu\cup\sigma,\mu\cup\sigma))+1-g(cohe_g(\mu\cap\sigma,\mu\cap\sigma))\leq 1-g(cohe_g(\mu,\mu))+1-g(cohe_g(\sigma,\sigma))$

iff

$g(cohe_g(\mu,\mu))+g(cohe_g(\sigma,\sigma))\leq g(cohe_g(\mu\cup\sigma,\mu\cup\sigma))+g(cohe_g(\mu\cap\sigma,\mu\cap\sigma))$.

Now, the immediate task is, given a N_g-ambiguity measure α_g, to extend this measure to a N_g- coherence measure. The result is as follows.

Theorem 5.2.

Let $\alpha_g:\mathcal{F}(X)\rightarrow[0,1]$ a N_g-ambiguity measure. Then $\alpha_g(\mu_{n_g})=n_g$ if and only if there exists a N_g-coherence measure $cohe_g$ such that $cohe_g(\mu,\mu)=N_g(\alpha_g(\mu))$ for all μ.

Proof. If $cohe_g(\mu,\mu)=N_g(\alpha_g(\mu))$ for all μ, in particular $cohe_g(\mu_{n_g},\mu_{n_g})=n_g=N_g(\alpha_g(\mu_{n_g}))$; then $\alpha_g(\mu_{n_g})=n_g$, and the sufficient condition is obtained.

Furthermore, if $\alpha_g(\mu_{n_g})=n_g$, we can define a $cohe_g:\mathcal{F}(X)\times\mathcal{F}(X)\rightarrow[0,1]$ by:

- $cohe_g(\emptyset,X)=0$
- $cohe_g(\mu_{n_g},\mu)=n_g$, for all μ
- $cohe_g(\mu,\mu)=N_g(\alpha_g(\mu))$ for all $\mu\in M_{g1}$
- For all $\mu,\sigma\in M_{g1}$, $\mu\neq\sigma$, $cohe_g(\mu,\sigma)=cohe_g(\sigma,\mu)=k\in[0,1]$
- For any $\mu\in M_{g1}$ and $\sigma\in M_{g2}$, $cohe_g(\mu,\sigma)=cohe_g(\sigma,\mu)=cohe_g(\mu,N_g(\sigma))$, defined above.
- And, finally, for any $\mu,\sigma\in M_{g2}$, $cohe_g(\mu,\sigma)=cohe_g(N_g(\mu),N_g(\sigma))$.

Of course, the $cohe_g$ defined thus is a N_g-coherence measure, and the desired conditions hold.

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