Possibility Relations: the hidden face of fuzzy preorders

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Abstract

The paper presents a new kind of fuzzy binary relations for modelling conditional possibility. The key idea is to consider fuzzy preorders as conditional necessity measures, and to obtain the associated conditional possibility measures afterwards by using a negation function.

Keywords: Fuzzy relations, fuzzy T-preorder, conditional possibility, conditional necessity.

1 Introduction

Since their introduction in [3], fuzzy preorders and similarity relations have generated a huge amount of literature. They appear under a variety of names depending on the authors, the involved fuzzy connectives (t-norms) or some minor changes performed to the standard definition. They have been the most commonly accepted way of extending the crisp notions of preorder and equivalence relation to the fuzzy framework.

Let us recall the definitions of both fuzzy relations. We will use the term *fuzzy equivalence relation* instead of the Zadeh's original *similarity relation*. The latter is usually reserved for those relations involving the t-norm $T = \min$. From now on, T will stand for a continuous t-norm.

Definition 1 *A binary fuzzy relation P on a set X, $P : X \times X \to [0,1]$, is a fuzzy T-preorder if for all x,y and z in X:*

(TN.1) $P(x,x) = 1$ *(reflexivity)*

(TN.2) $T(P(x,y), P(y,z))$ $\quad$ $P(x,z)$ *(T-transitivity)*

A fuzzy T-preorder E is a fuzzy T-equivalence iff for all x and y in X:

(TE.1) $E(x,y) = E(y,x)$ *(symmetry)*

Reflexivity and symmetry in the previous definitions look exactly the same as they do in the crisp case. Transitivity with respect to the t-norm T, or T-transitivity for short, is the natural generalisation of the crisp transitivity

if $P(x,y)$ *and* $P(y,z)$ *then* $P(x,z)$

when the t-norm T is chosen to play the role of the conjunctive particle *and* in the fuzzy domain.

If we think of the elements of X as facts that may either happen or not, $P(x,y) = \alpha$ can be then interpreted as a rule "*if x then y*" weighted with a certainty or necessity level α. It can be read as

the necessity of y given x is α

and it provides a measure of the strength of the conditional dependence of y upon x. Thus, the whole relation P becomes a kind of *conditional necessity*[1] *distribution* on pairs of elements of the set X.

Assuming then that a fuzzy preorder P has some sort of interpretation as conditional necessity measure, the question naturally arises on which the associated conditional possibility measure may be. Taking into account the relationship

[1] Remark that the term *necessity* is used here with, in principle, a more general meaning than that of standard Possibility Theory [1], where a necessity measure is required to be min-decomposable with respect to the conjunction connective.

between classical possibility and necessity measures, it seems rather natural to introduce a new fuzzy relation Q obtained from P by means of the condition

$$Q(x, y) = n(P(x, \neg y))$$

for all $x, y \in X$, where n stands for a strong negation function $n : [0, 1] \to [0, 1]$ and $\neg$ represents some sort of complement[2] $\neg : X \to X$. Relations introduced in the same way as Q will be called *fuzzy possibility relations*, and to them is devoted the present paper.

A meaning for $\neg$ is then introduced as follows. As long as the elements of X represent facts which may either happen or not, it is straightforward to think of them as binary variables taking values 0 or 1, the meaning of $x = 1$ being an occurrence of the fact x, whilst $x = 0$ standing for its lacking. A symptom that may either appear or not, a light that might be switched on or off, or a numeric variable taking a value above or below a threshold are examples of which sort of variables a particular element x in X provides represention for. Let us then identify every element x in X with the positive instantiation $x = 1$ of the associated variable, the meaning of $P(x, y) = \alpha$ being

if $x = 1$ *then* $y = 1$ *with necessity* α

It becomes now apparent the lack of expressiveness of the relational structure P in order to deal with rules involving negative instantiations of the variables, as for example

if $x = 0$ *then* $y = 1$ *with necessity* α

or

if $x = 0$ *then* $y = 0$ *with necessity* α

unless the set X is further enlarged with some new elements to represent these (lack of) occurrences. To do so, for every element x already in X a new one $\neg x$ has to be added such that represents the fact $x = 0$. Once this process is completed, a complement $\neg$ will be naturally defined on X.

Every fuzzy preorder on a set X will be seen from now on as a fragment -the positive part- of a conditional necessity distribution defined on a complemented set $\bar{X}$ containing X. Furthermore, given a negation function n every necessity relation P has exactly one associated possibility relation $Q(x, y) = n(P(x, \neg y))$ (and conversely) that convey exactly the same amount of information as P, although under a different apparence. The proposed meaning for $Q(x, y) = \alpha$ is a rule

y is possible given x

weighted with a level α, and can be read as

the possibility of y given x is α

It provides a measure on how possible it is that the two facts x and y coexist, or putting it in terms of instantiations of binary variables, that the two variables x and y take a simultaneous positive value $x = y = 1$.

2 T-necessity and S-possibility relations

In this section we will present an axiomatic definition of the fuzzy possibility relations we have just introduced, as well as some related elementary properties.

The starting point is the standard definition of fuzzy preorder, although fuzzy preorders will be referred to under a different name in the present context. We will call them *T-necessity relations*. The aim in doing so is to provide a clearer and compact system of naming through this paper, rather than replacing the traditional term beyond the scope of these pages.

In the following, as usual, T, S and n will stand for a continuous t-norm, a continuous t-conorm and a strong negation function repectively. Additionally, we shall require that T, S and n constitute a de Morgan triplet, i.e. $n(T(n(a), n(b))) = S(a, b)$ for all a, b in $[0, 1]$.

With Definition 1 in mind, let us rewrite (TN.1) and (TN.2) in terms of the associated relation Q defined by $Q(x, y) = n(P(x, \neg y)$. The following definition is then obtained.

Definition 2 *Let X be a complemented set. A fuzzy relation Q on a set X is an S-possibility if for all x,y and z in X:*

[2] From a merely formal point of view, we will choose the operator $\neg$ to be a complement, that is, a mapping $\neg : X \to X$ such that $\neg x \neq x$ and $\neg(\neg x) = x$ for all x in X.

(SP.1) $Q(x, \neg x) = 0$

(SP.2) $S(Q(x, \neg y), Q(y, z)) \geq Q(x, z)$ *(S-cotransitivity)*

The meaning of this newly introduced definition in the crisp case is quite obvious. Condition (SP.1) states that, given x, $\neg x$ is impossible. On the other hand, condition (SP.2) can be interpreted as a rule, namely

if *x makes z possible* **then** *x makes $\neg y$ possible or y makes z possible*

or also, by interpreting x, y and z as sets,

if $x \cap z \neq \emptyset$ **then** $x \cap \neg y \neq \emptyset$ *or* $y \cap z \neq \emptyset$

which shows that condition (SP.2) makes full sense in the crisp setting when thinking of possibility as a set theoretic intersection based measure.

An interesting feature of S-possibilities is that symmetry does not hold in general.

Definition 3 *Let X be a complemented set. A fuzzy relation on X is said to be counterpositive if $P(x, \neg y) = P(\neg x, y)$ for all x, y in X.*

Proposition 1 *Let P and Q be fuzzy relations on a complemented set X, related by $Q(x,y) = n(P(x, \neg y))$, for every x,y in X. Then*

(i) Q is symmetrical iff P is counterpositive,

(ii) P is symmetrical iff Q is counterpositive.

As a corollary, we have that symmetrical S-possibilities are obtained from counterpositive T-preorders, and also that S-possibilities obtained from fuzzy T-equivalence relations are counterpositive.

3 Representation theorems

Let us turn our attention to the problem of finding suitable representations for both T-necessity and S-possibility relations. As a matter of fact, T-necessity relations are nothing but fuzzy preorders, and a representation theorem for them is already known [2].

Theorem 1 (Representation for T-necessity relations) *A fuzzy relation P on a (complemented) set X is a T-necessity relation iff there exists a family H of fuzzy subsets of X such that*

$$P(x, y) = \inf_{h \in H} \hat{T}(h(x)|h(y))$$

for all x, y in X.

In this last expression, $\hat{T}$ stands for the residuum of T, i.e. $\hat{T}(a|b) = \sup\{c \in [0,1] \mid T(a,c)\ b\}$. Now, taking into account that $Q(x,y) = n(P(x, \neg y))$, it is possible to rewrite the previous representation as

$$n(Q(x, \neg y)) = \inf_{h \in H} \hat{T}(h(x)|h(y)).$$

We can then replace y by $\neg y$ to obtain the equivalent expression

$$\begin{aligned} Q(x,y) &= \sup_{h \in H} n(\hat{T}(h(x)|h(\neg y))) \\ &= \sup_{h \in H} \hat{S}(n(h(x))|n(h(\neg y))) \end{aligned}$$

which, in turn, provides a new representation theorem. Here $\hat{S}(a|b) = n(\hat{T}(n(a)|n(b)))$.

Theorem 2 (Representation for S-possibility relations) *A fuzzy relation Q on a complemented set X is an S-possibility relation iff there exists a family H of fuzzy subsets of X such that*

$$Q(x, y) = \sup_{h \in H} \hat{S}(n(h(x))|n(h(\neg y)))$$

for all x, y in X.

Note that the negation function becomes explicitly involved in the latter representation even though no reference to n at all has been made in the definition of S-possibility relations. This means that the family H depends on the chosen negation function n such that S, T and n constitute a de Morgan triplet.

It is also important to realise that the negation function n on the unit interval is by no means linked to the complementation $\neg$ on the set X. As a result of that, the expression $n(h(\neg y))$ admits no further simplification unless some new assumption is made as to the relationship between n

and $\neg$. Actually, it is more than a simple matter of algebraic simplification what is involved on the unboundness of n and $\neg$. Indeed, it is possible for a generator h to take values $h(x) = h(\neg x) = 1$, amongst others, making the representation rather counterintuitive.

Further research is needed in order to prove the existence of generating families such that $h(\neg x) = n(h(x))$ for every x in X. These will be called *strong generating families*, and the associated representations *strong representations*, taking the form $Q(x, y) = \sup_{h\in H} \hat{S}(n(h(x))|h(y))$ in the case of possibility relations.

Note that in the crisp case $\hat{S}(n(a)|b)$ is nothing but the classical truth function for the conjunction, and if T is a nilpotent t-norm, then $\hat{S}(n(a)|b) = T(a, b)$. Thus, the connective $\hat{S}(n(a)|b)$ can be seen as some sort of (generally) non-associative, non-commutative fuzzy conjunction, and the S-possibilities as conjunction-based possibility measures. Again, we are confronted to the fact that S-possibilities are non-symmetrical fuzzy relations in general.

4 Examples

The paper ends with two examples. The first one aims at showing that classical possibility and necessity measures are indeed particular cases of S-possibility and T-necessity relations. The second one presents possibility as a conjunctive based relation between fuzzy sets.

Example 1 Let X be a classical propositional language built from a finite set of propositional variables with connectives $\wedge$ and $\neg$. Other connectives like $\vee$ and $\rightarrow$ are defined as usual from $\wedge$ and $\neg$. Let - be the set of classical interpretations of X. Obviously X (modulo logical equivalence) is a complemented set with respect to $\neg$, since if $\varphi \in X$ then $\neg\varphi \in X$ and $\neg\neg\varphi$ is logically equivalent to φ. A possibility distribution on the set of interpretations $\pi : - \rightarrow [0, 1]$ induces a pair of dual possibility and necessity measures on propositions of X. Namely, for every $\varphi \in X$, one defines

$$\Pi(\varphi) = \max\{\pi(w) \mid w \in - , w \models \varphi\}$$
$$\text{and } N(\varphi) = 1 - \Pi(\neg\varphi).$$

Now we define the following binary fuzzy relations on X:

$$P(\varphi, \psi) = N(\neg\varphi \vee \psi), \quad Q(\varphi, \psi) = \Pi(\varphi \wedge \psi).$$

Then, from the well-known properties of possibility theory it is easy to check that P is a min-necessity relation and Q is a max-possibility relation. Furthermore, one can also check first that $T = \min$ and the family of mappings $H = \{h_w\}_{w\in}$, defined by $h_w(\varphi) = 1$ if $w \models \varphi$ and $h_w(\varphi) = 1 - \pi(w)$ otherwise, provide a representation of P in terms of Theorem 1. On the other hand, $S = \max$, the standard negation $n(a) = 1 - a$ and H provide a (non strong) representation of Q in terms of Theorem 2. Moreover in that case it holds the following simplification $\hat{S}(n(h_w(\varphi))|n(h_w(\neg\psi))) = \min(f_w(\varphi), f_w(\psi))$, where $f_w(\varphi) = 1 - h_w(\neg\varphi)$.

Example 2 Let $X = \{L_i\}_{i\in I}$ be a set of fuzzy subsets $L_i : U \rightarrow [0, 1]$ of a universe U, such that is complemented with respect to some negation function $N : [0, 1] \rightarrow [0, 1]$, i.e. for every i there exists exactly one $j \neq i$ such that $L_j(u) = N(L_i(u))$, for all $u \in U$. Denote such j by i_N. A complementation $\neg$ is then defined on X via $\neg L_i = L_{i_N}$. Consider the Lukasiewicz de Morgan triplet with $n(x) = 1 - x$ and define binary fuzzy relations on X as follows:

$$P(L_i, L_j) = \inf_{u\in U} \hat{T}(L_i(u)|L_j(u))$$

and $Q(L_i, L_j) = \sup_{u\in U} \hat{S}(1 - L_i(u)|L_j(u)) = T(L_i(u), L_j(u))$, where the last equality holds because T and S are the Lukasiewicz t-norm and t-conorm respectively. So defined, P and Q are a necessity-possibility pair (apply Ths. 1 and 2 with $H = U$ and $u(L_i) = L_i(u)$), and the representation provided by U is strong as long as $N(\alpha) = 1 - \alpha$, for all $\alpha \in [0, 1]$.

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