Stability of Aggregation Operators

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Abstract

Stability of aggregation operators is characterized and discussed. 1–stability turns to coincide with 1–Lipschitz property. Several examples and basic results are given.

Keywords: aggregation operator, Lipschitz property, norm, stability.

1 Introduction

Stability of any mathematical model of engineering problems means, roughly speaking, that the "small inputs errors" do not result to a "big output error". In the framework of aggregation operators, i.e., non–decreasing mappings $\bigcup_{n\in N}[0,1]^n \longrightarrow [0,1]$, $A(0,\dots,0)=0$, $A(1,\dots,1)=1$ and $A(x)=x$ for all $x\in[0,1]$, see [13], this desirable effect corresponds to the continuity of A. More, because of the compactness of $[0,1]^n$, $n\in N$, for any continuous aggregation operator A, $n\in N, \epsilon>0$ there is $\delta_{n,A,\epsilon}=\delta$ such that if the input errors do not exceed δ, the output error do not exceed ϵ, i.e., $\mid A(x_1,\dots,x_n)-A(y_1,\dots,y_n)\mid\leq\epsilon$ whenever $\max_i \mid x_i-y_i\mid\leq\delta$. For instance, in the case of the geometric mean G, i.e., $G(x,y)=\sqrt{xy}$, $(n=2)$ for a given $\epsilon>0$ the smallest relevant δ is equal to ϵ^2. However, for ensuring good accuracy of the result, say with $\epsilon=0.01$, we need to ensure extremely good accuracy of input values, $\delta=0.0001$ in our case, what is not realistic. To avoid such stability problems, we have to require stronger properties than simple continuity. The aim of this paper is the proposal of p–stability property of aggregation operators $p\in[1,\infty]$, and the discussion of p–stable aggregation operators. Several examples will be given.

2 p–stable aggregation operators

Recall the standard p–metric in $R^n, n\in N$, $p\in[1,\infty]$, given by $\|(x_1,\dots,x_n)\|_p=(\sum_{i=1}^{n}\mid x_i\mid^p)^{1/p}$ if $p>1$, and $\|(x_1,\dots,x_n)\|_\infty=\max_i\mid x_i\mid$.

Each n–ary aggregation operator $A_{(n)}:[0,1]^n\longrightarrow[0,1]$ can be understood as a mapping from a subspace of Banach space $(R^n,\|\cdot\|_p)$ to the Banach space $(R,\|\cdot\|_p)$ (observe that on R all metrics $\|\cdot\|_p$ coincide with the standard absolute value, $\|\cdot\|_p=\mid\cdot\mid$ for all $p\in[1,\infty]$). Then the norm of this mapping $\|A_{(n)}\|_p$ can be introduced as

$$\|A_{(n)}\|_p=\sup\left(\frac{\mid A_{(n)}(\mathbf{x})-A_{(n)}(\mathbf{y})\mid}{\|\mathbf{x}-\mathbf{y}\|_p}\right),$$

for all $\mathbf{x},\mathbf{y}\in[0,1]^n, \mathbf{x}\neq\mathbf{y}$. For the global aggregation operator $A:\bigcup_{n\in N}[0,1]^n\longrightarrow[0,1]$ the p–norm will be given by $\|A\|_p=\sup\Big(\|A_{(n)}\|_p\mid n\in N\Big)$. Evidently, $\|A_{(n)}\|_1=1$ for any aggregation operator A and hence $\|A\|_p\geq 1$.

Definition 1 *Let $A:\bigcup_{n\in N}[0,1]^n\longrightarrow[0,1]$ be an aggregation operator.*

1) *A will be called $n-p$-stable whenever $\|A_{(n)}\|_p\leq 1$.*

2) *A will be called p-stable if $\|A\|_p=1$, i.e., if A is $n-p$-stable for all $n\in N$.*

The class of all $n-p$-stable aggregation operators will be denoted as $\mathcal{S}_{n,p}$ while the class of all p-stable aggregation operators will be denoted as $\mathcal{S}_p$.

3 Basic properties of p–stable aggregation operators

Let A be a p-stable aggregation operator. Then for any $n \in N, \mathbf{x}, \mathbf{y} \in [0,1]^n$,

$$\begin{aligned} \mid A(x_1,\dots,x_n) - A(y_1,\dots,y_n) \mid &\le \\ \left(\sum_{i=1}^{n} \mid x_i - y_i \mid^p\right)^{\frac{1}{p}} &\le n^{\frac{1}{p}} \cdot \delta \quad (1) \end{aligned}$$

where $\delta = \max_i \mid x_i - y_i \mid$. Evidently, A is a uniformly continuous (and hence a continuous) aggregation operator. Further, to achieve an ε–accuracy of the output it is enough to ensure the $\frac{\varepsilon}{n^{\frac{1}{p}}}$–accuracy of the input. For two extremal cases $p = 1$ and $p = \infty$, p–stability means:

- the output error of a 1–stable aggregation operator A does not exceed the sum of input errors,
- the output error of a ∞–stable aggregation operator A does not exceed the maximal input error.

The choice of parameters $p \in [1, \infty]$ allows to control the output stability as given in (1). On the other hand, greater p smaller the possible choice of convenient aggregation operators. Indeed, because of the monotonicity of $\|\cdot\|_p$ in parameter p, $\mathcal{S}_p \supset \mathcal{S}_q$ whenever $p < q$ (similarly, for any $n \in N, n > 1$, $\mathcal{S}_{n,p} \supset \mathcal{S}_q$ for $p < q$), and $\mathcal{S}_q = \bigcap_{p<q} \mathcal{S}_p$ for all $q > 1$.

Note that for $p = 1$, 1–stable aggregation operators are exactly 1–Lipschitz aggregation operators which have been introduced and discussed in [2, 15]. For $n = 2$, $p \in \{1, \infty\}$, several interesting results can be found in [11]. Following [11], the weakest 1–stable aggregation operator is the Lukasiewicz t–norm T_L, $T_L(x_1,\dots,x_n) = \max\left(0, \sum_{i=1}^{n} x_i - (n-1)\right)$, and the strongest member of $\mathcal{S}_1$ is the Lukasiewicz t–conorm S_L, $S_L(x_1,\dots,x_n) = \min\left(1, \sum_{i=1}^{n} x_i\right)$.

Similarly it can be shown that the weakest and the strongest members of $\mathcal{S}_p$, $p \in [1, \infty]$, are the Yager t–norm T_p^Y and t–conorm S_p^Y, respectively, see [19, 9] where $T_\infty^Y = \min$, $S_\infty^Y = \max$, and for $1 \le p < \infty$,

$$\begin{aligned} T_p^Y(x_1,\dots,x_n) &= \max\left(0, 1 - \left(\sum_{i=1}^{n}(1-x_i)^p\right)^{\frac{1}{p}}\right) \quad S_p^Y(x_1,\dots,x_r \\ &= \max\left(0, 1 - \|1 - x\|_p\right). \end{aligned}$$

The next properties can be shown:

- If $A \in \mathcal{S}_p$ then $A^d \in \mathcal{S}_p$, where A^d is the dual aggregation operator of A, $A^d(x_1,\dots,x_n) = 1 - A(1-x_1,\dots,1-x_n)$. Observe that $S_p^Y = (T_p^Y)^d$ and that $(A^d)^d = A$,
- Let A be an aggregation operator and let $p \in [1, \infty]$. Then for any $m \in N$, $B_1,\dots,B_m \in \mathcal{S}_p$ (for any fixed $p \in [1,\infty]$), the composite aggregation operator $A(B_1,\dots,B_m) \in \mathcal{S}_p$ if and only if $A \in \mathcal{S}_\infty$,
- Each class $\mathcal{S}_p$ is a convex class closed in the uniform topology of $\bigcup_{n \in N} [0,1]^n \longrightarrow [0,1]$ mappings,
- If $A \in \mathcal{S}_\infty$ then A is idempotent.

4 Examples

Because of monotonicity of aggregation operators, $A \in \mathcal{S}_\infty$ if and only if A is shift–subinvariant, i.e., for all $a \in [0,1]$, $n \in N$, $x \in [0,1]^n$ such that $a + x \in [0,1]^n$ it is

$$A(a+x) \le a + A(x).$$

A typical example of a shift invariant aggregation operator is any Choquet–integral based operator. Therefore, the next classes are ∞–stable: $\mathcal{W}$ of all weighted means, $\mathcal{OWA}$ of all OWA operators [20], $\mathcal{WOWA}$ of WOWA operators [18]. Similarly, Sugeno and Shilkret integral produce ∞–stable aggregation operators.

Evidently, for an associative operator A, $A \in \mathcal{S}_\infty$ if and only if $A_{(2)} \in \mathcal{S}_{2,\infty}$. This observation allows to check that k–medians med_k [7] are also ∞–stable operators. Observe that a charecterization of all (binary) shift invariant aggregation operators can be found in [12].

In the framework of triangular norms (and then by duality also of t–conorms), p-stability ensures the continuity of T, which is then necessarily an ordinal sum ($< a_k, b_k, T_k >$) with continuous Archimedean summands (for more details we recommend [9]). However, for $p = \infty$, the idempotency of all ∞–stable aggregation operators means that the unique ∞–stable t–norm is the min. On the other hand, for $p = 1$, due to [14], see also [17, 16, 9], a t–norm T is 1–stable if and only if it is and ordinal sum of summands T_k with convex additive generator f_k. Similarly for $p \in]1, \infty[$, we can show that $T \in \mathcal{S}_p$ if and only if all T_k from its ordinal sum representation are p–stable. For a continuous Archimedean t–norm T_k we conjecture that $T_k \in \mathcal{S}_p$, $p \in]1, \infty[$ if and only if $f_k^{\frac{1}{p}}$ is convex.

From the other well known classes of aggregation operators, note that the root–power operators M_λ [5], $M_\lambda \in \mathcal{S}_\infty$ for all $\lambda \in [1, \infty] \cup \{-\infty\}$ while $M_\lambda \notin \mathcal{S}_1$ for all $\lambda \in] - \infty, 1[$.

Because of the non–continuity of uninorms [6, 8], uninorms cannot be p–stable for any $p \in [1, \infty]$. However, for nullnorms we have the next interesting result: following [2] each nullnorm V can be defined as a composite operator $V = med_a(T, S)$, where a is the annihilator of V, T is some t-norm and S is some t-conorm. Therefore, $V \in \mathcal{S}_p$ if and only if there are $T, S \in \mathcal{S}_p$ such that $V = med_a(T, S)$.

5 Acknowledgements

The work of the first author is partially supported by the projects BFN2000–1114, TIC2000–1368–C03–01, E007/2001 while the second author is supported by the grants VEGA 1/8331/01, 1/7076/20 and GACR 402/99/0032.

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