Towards Studying of Fuzzy Information Relations

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Abstract

Information relations represent relationships among objects determined by properties of these objects. In this paper we consider a fuzzy generalization of these relations. Several classes of these relations are defined and their basic properties are given.

Keywords: Information Relations, Fuzzy Relations Fuzzy Logical Operators.

1 Introduction

In many application domains data have the form of a collection of objects together with descriptions of these objects, usually representing properties of objects. From descriptions of objects, often referred to as explicit information, we can derive relationships among these objects. These relationships constitute implicit information contained in user's data and reflect some aspects of incompleteness of explicit information. Formally, they are binary relations on a domain of objects such that every relation is determined by a set of properties of the respective objects (we actually deal with binary relations parameterized by sets of properties of objects). They are called *information relations* representing either *indistinguishabilities* of objects (characterizing some kinds of "sameness" of objects) or *distinguishabilities* (reflecting some types of differences among objects). Information relations were extensively investigated in the literature (see [1],[2],[3],[4],[5],[11]), mainly in the context of *information logics* – logical systems capable to derive information about relations between objects determined by the properties of these objects.

While some properties of objects naturally correspond to two–valued notions (e.g. Name(s)) and can be naturally represented by *crisp* structures, others are *fuzzy* in their nature – consequently, available information is often imprecise also. For example, when a database contains the property Speaking foreign language, the meaningful information is to what extent a person x speaks a language L, and it seems a far-going simplification to distinguish only two categories: x speaks L or not. Furthermore, when the descriptions of objects are fuzzy, the relations between these objects are to be fuzzy as well. Assume, for example, that a database contains information about *Alan*, *Jim* and *Tom*; they speak English *fluently*, *quite fluently* and *weak*, respectively. The natural conclusion is that *Alan* is "more similar" to *Jim* than to *Tom* with respect to their ability of speaking English.

Clearly, fuzzy information cannot be adequately represented by standard methods based on two–valued structures. A natural solution seems to be fuzzy generalizations of the respective methods.

Recently, Orłowska ([6]) has proposed a general framework for generalizing information logics for the multi–valued case. In the present paper we extend this approach. The notion of information system is generalized for the case where properties of objects are assumed to be fuzzy sets in the respective domains. We consider several binary fuzzy relations between fuzzy sets. On the basis of these relations, some classes of fuzzy information relations are defined and their basic properties are given.

2 Preliminaries

Throughout this paper we will write $\mathcal{T}$, $\mathcal{S}$, $\mathcal{J}$ and $\mathcal{N}$ to denote a triangular norm (t–norm), a triangular conorm (t–conorm), an implicator and a negator, respectively. Due to associativity and commutativity of t–norms (resp. t–conorms), for any natural number n and $\mathcal{A}=\{x_1,\dots,x_n\}\subseteq[0,1]$, we will write $\mathcal{T}_{x\in\mathcal{A}}(x)$ (resp. $\mathcal{S}_{x\in\mathcal{A}}(x)$) to denote

$$\mathcal{T}_{x\in\mathcal{A}}(x)=\mathcal{T}(x_1,\dots,x_n)$$
$$\mathcal{S}_{x\in\mathcal{A}}(x)=\mathcal{S}(x_1,\dots,x_n)$$

for $n>0$ and $\mathcal{T}_{x\in\mathcal{A}}(x)=1$ (resp. $\mathcal{S}_{x\in\mathcal{A}}(x)=0$) for $n=0$. A point $x_0\in(0,1)$ is a *zero divisor* of a t–norm $\mathcal{T}$ iff $\mathcal{T}(x_0,y_0)=0$ for some $y_0\in(0,1)$. By T^+ we denote the class of all t–norms without zero divisors. Recall that for a left–continuous t–norm $\mathcal{T}$, its residuum is defined by:

$$\mathcal{J}^{\mathcal{T}}(x,y)=\sup\{\lambda\in[0,1]:\mathcal{T}(x,\lambda)\leqslant y\}.$$

For a left–continuous t–norm $\mathcal{T}$, we will write $\mathcal{N}^{\mathcal{T}}$ to denote the negator induced by $\mathcal{J}^{\mathcal{T}}$, i.e. $\mathcal{N}^{\mathcal{T}}(x)=\mathcal{J}^{\mathcal{T}}(x,0)$, $x\in[0,1]$.
Given a negator $\mathcal{N}$ and $A\in\mathcal{F}(\mathfrak{X})$, we will write $co_{\mathcal{N}}A$ to denote the $\mathcal{N}$–complement of A, i.e. the fuzzy set $co_{\mathcal{N}}A(x)=\mathcal{N}(A(x))$, $x\in\mathfrak{X}$.

Let $\mathcal{T}$ and $\mathcal{N}$ be a t–norm and a negator, respectively, and let n be a natural number. A binary fuzzy relation R on $\mathfrak{X}$ is called:

- *reflexive* iff $R(x,x)=1$ for all $x\in\mathfrak{X}$
- *pseudo–reflexive* iff $R(x,x)>0$ for all $x\in\mathfrak{X}$
- *quasi–reflexive* iff $\sup_{y\in\mathfrak{X}}R(x,y)>0$ implies $R(x,x)>0$ for all $x\in\mathfrak{X}$
- *irreflexive* iff $R(x,x)=0$ for all $x\in\mathfrak{X}$
- *weakly irreflexive* iff $R(x,x)=\inf_{y\in\mathfrak{X}}R(x,y)$ for all $x\in\mathfrak{X}$
- *symmetric* iff $R(x,y)=R(y,x)$ for all $x,y\in\mathfrak{X}$
- $\mathcal{T}^n$*–transitive* iff for every $x,x_1,\dots,x_n,y\in\mathfrak{X}$ $\mathcal{T}(R(x,x_1),\dots,R(x_n,y))\leqslant R(x,y)$
- $(\mathcal{T}^n,\mathcal{N})$*–cotransitive* iff $co_{\mathcal{N}}R$ is $\mathcal{T}^n$–transitive.

A $\mathcal{T}^1$–transitive relation is just $\mathcal{T}$–transitive; $(\mathcal{T}^n,\mathcal{N}^{\mathcal{T}})$–cotransitive relations will be called $\mathcal{T}^n$–cotransitive.

If R is reflexive and symmetric then it is called a *tolerance* relation; for a t–norm $\mathcal{T}$, a $\mathcal{T}$–transitive tolerance relation is called $\mathcal{T}$–equivalence relations. A $\mathcal{T}$–equivalence relation R satisfying the separation property: $R(x,y)=1\Leftrightarrow x=y$, is called a $\mathcal{T}$*–equality*.

3 Some Binary Fuzzy Relations on $\mathcal{F}(\mathfrak{X})$

In this section we will consider some fuzzy relations between two fuzzy sets in $\mathfrak{X}$. We pay particular attention to fuzzy relations measuring degrees to which $A\in\mathcal{F}(\mathfrak{X})$ and $B\in\mathcal{F}(\mathfrak{X})$ are either *indistinguishable* or *distinguishable*.[1]

Let $\mathcal{T}$ and $\mathcal{J}$ be a t–norm and an implicator, respectively. The following binary fuzzy relations on $\mathcal{F}(\mathfrak{X})$, called ***J*–inclusion** and ***T*–compatibility**, are defined by: for every $A,B\in\mathcal{F}(\mathfrak{X})$

$$\begin{aligned}Inc_I(A,B)&=\inf_{y\in\mathfrak{X}}\mathcal{J}(A(x),B(x))\\ Com_{\mathcal{T}}(A,B)&=\sup_{y\in\mathfrak{X}}\mathcal{T}(A(x),B(x)).\end{aligned}$$

Intuitively, $Inc_I(A,B)$ (resp. $Com_{\mathcal{T}}(A,B)$) is the degree to which A is included in B (resp. A and B overlap).

Let $\mathcal{T}$ be a left–continuous t–norm, $\mathcal{J}$ be its residuum, $\mathcal{N}=\mathcal{N}^{\mathcal{T}}$ and let $\mathcal{S}$ be a t–conorm. Define the following binary fuzzy relations on $\mathcal{F}(\mathfrak{X})$: for every $A,B\in\mathcal{F}(\mathfrak{X})$,

- ***T*–orthogonality**:
$$Ort_{\mathcal{T}}(A,B)=Inc_{\mathcal{T}}(A,co_{\mathcal{N}}B)$$
- ***T*–indiscernability**:
$$Ind_{\mathcal{T}}(A,B)=\mathcal{T}(Inc_I(A,B),Inc_I(B,A))$$
- **$(\mathcal{T},\mathcal{S})$–diversity**:
$$Div_{\mathcal{T},\mathcal{S}}(A,B)=\mathcal{S}(Com_{\mathcal{T}}(A,co_{\mathcal{N}}B),Com_{\mathcal{T}}(co_{\mathcal{N}}A,B))$$
- ***T*–complementarity**:
$$Cmp_{\mathcal{T}}(A,B)=Ind_{\mathcal{T}}(A,co_{\mathcal{N}}B)$$
- **$(\mathcal{T},\mathcal{S})$–incomplementarity**:
$$Icm_{\mathcal{T},\mathcal{S}}(A,B)=\mathcal{S}(Com_{\mathcal{T}}(A,B),Com_{\mathcal{T}}(co_{\mathcal{N}}A,co_{\mathcal{N}}B)).$$

$Ort_{\mathcal{T}}(A,B)$ represents the degree to which fuzzy sets A and B are disjoint. $Ind_{\mathcal{T}}(A,B)$ ($Div_{\mathcal{T},\mathcal{S}}(A,B)$) is the degree to which fuzzy sets A and B are indiscernible (different), whereas $Cmp_{\mathcal{T}}(A,B)$ ($Icm_{\mathcal{T},\mathcal{S}}(A,B)$) is the degree to which A is the $\mathcal{N}^{\mathcal{T}}$–complement of B (A differs from the $\mathcal{N}^{\mathcal{T}}$–complement of B).

[1] $\mathcal{F}(\mathfrak{X})$ stands for the family of all fuzzy sets in $\mathfrak{X}$.

The formulation of $Div_{\mathcal{T},\mathcal{S}}(A,B)$ and $Icm_{\mathcal{T},\mathcal{S}}(A,B)$ are inspired by the following (crisp) equivalence: for any $A,B \subseteq \mathfrak{X}$,

$$A \neq B \iff ((\exists x \in \mathfrak{X})\,(x \in A \wedge x \notin B)) \vee ((\exists x \in \mathfrak{X})\,(x \notin A \wedge x \in B)).$$

Proposition 1 *Let $\mathcal{T}$ be a left–continuous t–norm, $\mathcal{J}$ be its residuum, $\mathcal{N}=\mathcal{N}^{\mathcal{T}}$ and let $\mathcal{S}$ be a t–conorm. Then*

- *$Inc_{\mathcal{J}}$ is reflexive and $\mathcal{T}$–transitive*
- *$Com_{\mathcal{T}}$ is symmetric and for $\mathcal{T} \in T^{+}$, quasi–reflexive*
- *$Ort_{\mathcal{T}}$ is symmetric and for $\mathcal{T} \in T^{+}$, weakly irreflexive*
- *$Ind_{\mathcal{T}}$ is a $\mathcal{T}$–equality*
- *$Div_{\mathcal{T},\mathcal{S}}$ is irreflexive, symmetric and for $\mathcal{T} \in T^{+}$, $\mathcal{T}$–cotransitive*
- *$Cmp_{\mathcal{T}}$ is $\mathcal{T}^2$–transitive and for $\mathcal{T} \in T^{+}$, irreflexive*
- *$Icm_{\mathcal{T},\mathcal{S}}$ is symmetric and for $\mathcal{T} \in T^{+}$, pseudo–reflexive and $\mathcal{T}^2$- cotransitive.* □

4 Fuzzy Information Relations

By a ***fuzzy information system*** we mean a tuple $\Sigma=(OB,AT,\{V_a : a \in AT\})$, where $OB \neq \emptyset$ is a set of objects and $AT \neq \emptyset$ is a finite set of attributes; each $A \in AT$ is a mapping $a : OB \to \mathcal{F}(V_a)$.

Intuitively, for every object $x \in OB$, every attribute $a \in AT$ and every $v \in V_a$, $a(x)(v)$ is the degree to which the object x has the value v on the attribute a.

Let $\mathcal{T}$ be a left–continuous t–norm, $\mathcal{J}$ be its residuum, $\mathcal{N}=\mathcal{N}^{\mathcal{T}}$ and let $\mathcal{S}$ be a t–conorm. For any fuzzy information system $\Sigma=(OB,AT,\{V_a : a \in AT\})$ and any $\mathcal{A} \subseteq AT$, let us define the following classes of binary fuzzy relations on $\mathcal{F}(OB)$ in Σ, called ***fuzzy information relations in Σ induced by $\mathcal{A}$***: for every $x,y \in OB$,

A. Fuzzy indistinguishability relations:

1. strong (weak) $\mathcal{T}$–indiscernability relations:

$$\mathsf{ind}^{s}_{\mathcal{T}}(\mathcal{A})(x,y) = \mathcal{T}_{a \in \mathcal{A}}(Ind_{\mathcal{T}}(a(x),a(y)))$$
$$\mathsf{ind}^{w}_{\mathcal{T}}(A)(x,y) = \mathcal{S}_{a \in \mathcal{A}}(Ind_{\mathcal{T}}(a(x),a(y)))$$

2. strong (weak) $\mathcal{T}$–compatibility relations:

$$\mathsf{com}^{s}_{\mathcal{T}}(\mathcal{A})(x,y) = \mathcal{T}_{a \in \mathcal{A}}(Com_{\mathcal{T}}(a(x),a(y)))$$
$$\mathsf{com}^{w}_{\mathcal{T}}(\mathcal{A})(x,y) = \mathcal{S}_{a \in \mathcal{A}}(Com_{\mathcal{T}}(a(x),a(y)))$$

3. strong (weak) $(\mathcal{T},\mathcal{S})$–incomplementarity relations:

$$\mathsf{icm}^{s}_{\mathcal{T},\mathcal{S}}(\mathcal{A})(x,y) = \mathcal{T}_{a \in \mathcal{A}}(Icm_{\mathcal{T},\mathcal{S}}(a(x),a(y)))$$
$$\mathsf{icm}^{w}_{\mathcal{T},\mathcal{S}}(\mathcal{A})(x,y) = \mathcal{S}_{a \in \mathcal{A}}(Icm_{\mathcal{T},\mathcal{S}}(a(x),a(y)))$$

B. Fuzzy distinguishability relations:

1. strong (weak) $(\mathcal{T},\mathcal{S})$–diversity relations:

$$\mathsf{div}^{s}_{\mathcal{T},\mathcal{S}}(\mathcal{A})(x,y) = \mathcal{T}_{a \in \mathcal{A}}(Div_{\mathcal{T},\mathcal{S}}(a(x),a(y)))$$
$$\mathsf{div}^{w}_{\mathcal{T},\mathcal{S}}(\mathcal{A})(x,y) = \mathcal{S}_{a \in \mathcal{A}}(Div_{\mathcal{T},\mathcal{S}}(a(x),a(y)))$$

2. strong (weak) $\mathcal{T}$–orthogonality relations:

$$\mathsf{ort}^{s}_{\mathcal{T}}(\mathcal{A})(x,y) = \mathcal{T}_{a \in \mathcal{A}}(Ort_{\mathcal{T}}(a(x),a(y)))$$
$$\mathsf{ort}^{w}_{\mathcal{T}}(\mathcal{A})(x,y) = \mathcal{S}_{a \in \mathcal{A}}(Ort_{\mathcal{T}}(a(x),a(y)))$$

3. strong (weak) $\mathcal{T}$–complementarity relations:

$$\mathsf{cmp}^{s}_{\mathcal{T}}(\mathcal{A})(x,y) = \mathcal{T}_{a \in \mathcal{A}}(Cmp_{\mathcal{T}}(a(x),a(y)))$$
$$\mathsf{cmp}^{w}_{\mathcal{T}}(\mathcal{A})(x,y) = \mathcal{S}_{a \in \mathcal{A}}(Cmp_{\mathcal{T}}(a(x),a(y))).$$

Note that for every Σ, every fuzzy information relation irel and every $a \in AT$, $\mathsf{irel}^{s}(\{a\}) = \mathsf{irel}^{w}(\{a\})$. Also, $\mathsf{irel}^{s}(\emptyset) = \mathfrak{X} \times \mathfrak{X}$ and $\mathsf{irel}^{w}(\emptyset) = \emptyset$.

Intuitively, for $\Sigma=(OB,AT,\{V_a : a \in AT\})$, $\mathcal{A} \subseteq AT$, a fuzzy information relation $\mathsf{irel}(\mathcal{A})$ and $x,y \in OB$, $\mathsf{irel}^{s}(\mathcal{A})(x,y)$ (resp. $\mathsf{irel}^{w}(\mathcal{A})(x,y)$ is the degree to which x is irel–related with y for all (resp. some) $a \in \mathcal{A}$. In particular, $\mathsf{ind}(\mathcal{A})(x,y)$ represents the degree to which for all attributes $a \in \mathcal{A}$, characteristics $a(x)$ and $a(y)$ of x and y, respectively, are $\mathcal{T}$–equal.

Important applications of fuzzy indistinguishability relations concern the representation of approximations of fuzzy sets $F \in \mathcal{F}(OB)$ in fuzzy information systems. Specifically, for a left–continuous t–norm $\mathcal{T}$, its residuum $\mathcal{J}$, any $\mathcal{A} \subseteq AT$, any fuzzy information relation $\mathsf{irel}(\mathcal{A})$ and any $F \in \mathcal{F}(OB)$, let us define an *$\mathcal{J}$–lower* and a *$\mathcal{T}$–upper fuzzy approximation of F wrt* $\mathsf{irel}_{\mathcal{T}}(\mathcal{A})$ by: for every $x \in OB$,

$$\underline{\mathsf{irel}(\mathcal{A})}_{\mathcal{J}}(F)(x) = \inf_{y \in OB} \mathcal{J}(\mathsf{irel}_{\mathcal{T}}(\mathcal{A})(x,y),F(y))$$
$$\overline{\mathsf{irel}(\mathcal{A})}^{\mathcal{T}}(F)(x) = \sup_{y \in OB} \mathcal{T}(\mathsf{irel}_{\mathcal{T}}(\mathcal{A})(x,y),F(y)).$$

For example, let F represent an expert decision σ (i.e. $F(x)$ is the degree to which the object x coincides with σ). Then $\underline{\mathsf{irel}(\mathcal{A})}_{\mathcal{J}}(F)(x)$ (resp. $\overline{\mathsf{irel}(\mathcal{A})}^{\mathcal{T}}(F)(x)$) might be viewed as the degree to which x *certainly* (resp. *possibly*) coincides with σ.

In the fuzzy rough sets theory[2] settings, for $\mathsf{irel} = \mathsf{ind}^s_{\mathcal{T}}$ we have the following hierarchy of *$\mathcal{T}$–definability*, namely: any $F \in \mathcal{F}(OB)$ is called

- *totally $\mathcal{T}$-definable wrt $\mathcal{A}$* iff $\underline{\mathsf{irel}(\mathcal{A})}_{\mathcal{J}}(F) = \overline{\mathsf{irel}(\mathcal{A})}^{\mathcal{T}}(F)$
- *roughly $\mathcal{T}$–definable wrt $\mathcal{A}$* iff $\underline{\mathsf{irel}(\mathcal{A})}_{\mathcal{J}}(F) \neq \emptyset$ and $\overline{\mathsf{irel}(\mathcal{A})}^{\mathcal{T}}(F) \neq OB$
- *$\mathcal{T}$–indefinable wrt $\mathcal{A}$* iff $\underline{\mathsf{irel}(\mathcal{A})}_{\mathcal{J}}(F) = \emptyset$ and $\overline{\mathsf{irel}(\mathcal{A})}^{\mathcal{T}}(F) = OB$.

The following theorem provides basic properties of $\mathcal{T}$–information relations.

Theorem 1 *Let $\mathcal{T}$ be a left–continuous t–norm and let $\Sigma = (OB, AT, \{V_a : a \in AT\})$ be a fuzzy information system. Then for every $\mathcal{A} \subseteq AT$ and every $a \in AT$,*

1. (a) $\mathsf{ind}^s_{\mathcal{T}}(\mathcal{A})$ *are $\mathcal{T}$–equivalence relations*

(b) *for $\mathcal{A} \neq \emptyset$, $\mathsf{ind}^w_{\mathcal{T}}(\mathcal{A})$ are fuzzy tolerance relations* $\mathsf{ind}^w_{\mathcal{T}}(\{a\})$ *are $\mathcal{T}$–equivalence relations*

2. $\mathsf{com}^s_{\mathcal{T}}(\mathcal{A})$ *and* $\mathsf{com}^w_{\mathcal{T}}(\mathcal{A})$ *are symmetric and for $\mathcal{T} \in T^+$, quasi–reflexive*

3. (a) $\mathsf{icm}^s_{\mathcal{T},\mathcal{S}}(\mathcal{A})$ *is symmetric*
for $\mathcal{T} \in T^+$, $\mathsf{icm}^s_{\mathcal{T},\mathcal{S}}(\mathcal{A})$ *are reflexive and* $\mathsf{icm}^s_{\mathcal{T},\mathcal{S}}(\{a\})$ *are $\mathcal{T}^2$–cotransitive*

(b) $\mathsf{icm}^w_{\mathcal{T},\mathcal{S}}(\mathcal{A})$ *are symmetric*
for $\mathcal{T} \in T^+$, $\mathsf{icm}^w_{\mathcal{T},\mathcal{S}}(\mathcal{A})$ *are $\mathcal{T}^2$–cotransitive and for $\mathcal{A} \neq \emptyset$, reflexive*

4. (a) $\mathsf{div}^s_{\mathcal{T}\mathcal{S}}(\mathcal{A})$ *are symmetric*
for $\mathcal{A} \neq \emptyset$, $\mathsf{div}^s_{\mathcal{T},\mathcal{S}}(\mathcal{A})$ *are irreflexive*
for $\mathcal{T} \in T^+$, $\mathsf{div}^s_{\mathcal{T},\mathcal{S}}(\{a\})$ *are $\mathcal{T}$–cotransitive*

(b) $\mathsf{div}^w_{\mathcal{T},\mathcal{S}}(\mathcal{A})$ *are symmetric, irreflexive and for $\mathcal{T} \in T^+$, $\mathcal{T}$–cotransitive*

5. $\mathsf{ort}^s_{\mathcal{T}}(\mathcal{A})$ *and* $\mathsf{ort}^w_{\mathcal{T}}(\mathcal{A})$ *are symmetric and for $\mathcal{T} \in T^+$, weakly irreflexive*

6. $\mathsf{cmp}^s_{\mathcal{T}}(\{a\})$ *and* $\mathsf{cmp}^w_{\mathcal{T}}(\mathcal{A})$ *are $\mathcal{T}^2$–transitive for $\mathcal{T} \in T^+$,* $\mathsf{cmp}^s_{\mathcal{T}}(\mathcal{A})$ *(for $\mathcal{A} \neq \emptyset$) and* $\mathsf{cmp}^w_{\mathcal{T}}(\mathcal{A})$ *are irreflexive.* □

Fuzzy information relations are currently investigated in the context of fuzzy information logics – formalisms capable to derive conclusions on (fuzzy) relationships among objects in fuzzy information systems. Preliminary results are presented in [10].

[2]Fuzzy rough sets were broadly investigated in [7],[8] and [9].

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