ON FUZZY ORDERINGS OF CRISP AND FUZZY INTERVALS

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First, fuzzy membership functions for the assertion '[x,y] is a positive interval' are proposed and characterized via non-decreasing real maps. Last, using those functions together with the interval-difference and the notion of average index, comparison indexes between intervals and the ones between fuzzy intervals are proposed.

1 Introduction

In the literature, there are a lot of methods concerning the problem of ordering the interval numbers $[\underline{x},\overline{x}],[\underline{y},\overline{y}],...$ ([1], [7]) or the fuzzy ones $A,B,...$ (see [4], [9] and [5] for an overview of methods). All of these methods can be classified as belonging to two different approaches. (i) Ordering the crisp or the fuzzy intervals using binary order relations: $[\underline{x},\overline{x}] \prec [\underline{y},\overline{y}]$, $A \preceq B,....$ (ii) Giving comparison indexes $R([\underline{x},\overline{x}],[\underline{y},\overline{y}])$ or $R(A,B)$ in $[0,1]$.

In relation with previous works ([6], [2]), the present paper deals with ordering procedures of interval numbers or fuzzy quantities belonging to the aforementioned second class. Specifically, we propose a theoretical approach to the comparison of pairs of the approximate measurements $[\underline{x},\overline{x}],[\underline{y},\overline{y}],...$ or pairs of fuzzy quantities $A,B,...$ in the following way:

(a) The crisp and fuzzy intervals are considered as points belonging to crisp subsets. (b) Using fuzzy extensions of the characteristic function $f_{\mathbb{R}^+}$ and arithmetic interval operations, comparison indexes $R([\underline{x},\overline{x}],[\underline{y},\overline{y}]) \in [0,1]$ between intervals and the ones $R(A,B) \in [0,1]$ between fuzzy intervals are defined.

The paper is structured as follows. First, some necessary basic results on the crisp and fuzzy Interval Analysis fields are presented. Second, fuzzy membership functions (denoted by Pos) for the assertion '$[\underline{x},\overline{x}]$ is a positive interval' are proposed. Third, we characterize every aforementioned function Pos by a real non-decreasing map $g_{Pos}: [0,1] \longrightarrow [0,\frac{1}{2}]$. Fourth, using a map Pos and the *interval-difference* $[y]-[x] = [\underline{y}-\overline{x},\overline{y}-\underline{x}]$, the associated comparison index R_{Pos} as a fuzzy relation in $\mathfrak{I}(\mathbb{R})$ is defined. Some properties of the maps type R_{Pos} are analysed. Finally, using average indexes (see [3]), extensions of the previous maps Pos and R_{Pos} to a class $\mathcal{F}\mathfrak{I}(\mathbb{R})$ of *fuzzy intervals* are defined. Properties and examples are included.

2 Basic notions and notations

In this section we recall some basic arithmetic operations, results and binary relations related to the crisp and fuzzy Interval Analysis.

2.1 Some results on Interval Analysis

In the field of *Interval Analysis* ([8], [7]) an interval number can be thought as an extension of the concept of a real number. If $\mathfrak{I}(\mathbb{R})$ denotes the set of *interval numbers*, we consider $\mathbb{R}$ as a proper subset of $\mathfrak{I}(\mathbb{R})$ identifying $x \in \mathbb{R}$ with $[x,x] = \{x\}$. In this way, $[x] = [\underline{x},\overline{x}],[y] = [\underline{y},\overline{y}],...$ denote generic interval numbers of $\Im(\mathbb{R})$ and $x, k, \ldots$ without distinction symbolize real numbers or degenerate intervals $[x, x], [k, k], \ldots$ in $\Im(\mathbb{R})$.

The symbol $\preccurlyeq$ denotes the crisp order in $\Im(\mathbb{R})$: $[x] \preccurlyeq [y] \Longleftrightarrow (\underline{x} \leqq \underline{y}) \& (\overline{x} \leqq \overline{y})$.

As well, the next usual operations of Interval Arithmetic [8] give extensions of the same operations in $\mathbb{R}$: $-[x] = [-\overline{x}, -\underline{x}]$, $[x] + [y] = [\underline{x}+\underline{y}, \overline{x}+\overline{y}]$, $[y] - [x] = [y] + (-[x]) = [\underline{y} - \overline{x}, \overline{y} - \underline{x}]$

and (if $k \in \mathbb{R}^+$) $k[x] = [k\underline{x}, k\overline{x}]$.

The expressions x^+ and x^- denote the positive and negative parts of $x \in \mathbb{R}$: $x^+ = \max(0, x)$ and $x^- = (-x)^+$.

An interval number $[x] = [\underline{x}, \overline{x}]$ is said to be *positive* [7] if $\underline{x} \geq 0$, *strictly positive* if $\underline{x} > 0$ (or $\underline{x}^+ \neq 0$), *negative* if $\overline{x} \leqq 0$ and *strictly negative* if $\overline{x} < 0$ (or $\overline{x}^- \neq 0$).

2.2 *Fuzzy Interval Analysis*

$\mathcal{F}(\mathbb{R})$ denotes a set of fuzzy quantities [9]. In the following, a fuzzy quantity is a normalized fuzzy subset A of $\mathbb{R}$ with bounded support $supp(A)$ verifying also: (a) For all $\alpha \in (0,1]$, the α-cut $A_\alpha \subset \mathbb{R}$ is an interval. (b) If $Cl(supp(A))$ denotes the topological closure, then the restriction $A_{/Cl(supp(A))}$ is a continuous map. In consequence, if A belongs to $\mathcal{F}\Im(\mathbb{R})$ then $Cl(supp(A))$ and the α-cuts A_α ($\alpha \neq 0$) are closed intervals. The class $\Im(\mathbb{R})$ of the interval numbers $[x] = [\underline{x}, \overline{x}]$ can be viewed as a proper subset of $\mathcal{F}\Im(\mathbb{R})$. When we express an interval as a fuzzy quantity, we should usually retain the simpler non fuzzy interval notation $[x]$. With this criteria we can write: $\mathbb{R} \subset \Im(\mathbb{R}) \subset \mathcal{F}\Im(\mathbb{R})$.

The following arithmetic operations in $\mathcal{F}\Im(\mathbb{R})$ are extensions of the ones in $\Im(\mathbb{R})$:

$$(-A)_\alpha = [-\overline{a_\alpha}, -\underline{a_\alpha}],$$

$$(A+B)_\alpha = [\underline{a_\alpha} + \underline{b_\alpha}, \overline{a_\alpha} + \overline{b_\alpha}] \;,$$

$$(B-A)_\alpha = (B+(-A))_\alpha = [\underline{b_\alpha} - \overline{a_\alpha}, \overline{b_\alpha} - \underline{a_\alpha}]$$

$$\text{and (if } k \geq 0), (kA)_\alpha = [k\underline{a_\alpha}, k\overline{a_\alpha}].$$

Finally, we take into account the following binary relation $\preccurlyeq$ in $\mathcal{F}\Im(\mathbb{R})$:

$$A \preccurlyeq B \Longleftrightarrow A_\alpha \preccurlyeq B_\alpha \quad \forall \alpha \in (0,1]$$

3 Vague ordering of the crisp and fuzzy interval numbers

First, positivity indicators Pos on interval numbers are defined and characterized. Next, the associated concepts $Neg, SPos$ and $SNeg$ are defined.

3.1 *Vague positivity indicators on the set $\Im(\mathbb{R})$ of interval numbers*

A definition of positivity indicator as a fuzzy subset is proposed.

Definition 1 *A map Pos: $\Im(\mathbb{R}) \longrightarrow [0,1]$ is a positivity indicator on $\Im(\mathbb{R})$ iff:*

(P1) $Pos(x) = 1 \; if \; x \geq 0$ and $Pos(x) = 0 \; if \; x < 0$

(P2) $[x] \preccurlyeq [y] \Longrightarrow Pos([x]) \leqq Pos([y])$

(P3) $Pos(k[x]) = Pos([x]) \; \forall k > 0 \; \forall [x] \in \Im(\mathbb{R})$

(P4) *If $\underline{x} \neq 0$ and $\overline{x} \neq 0$, then* $Pos(-[x]) = 1 - Pos([x])$

Properties and a characterization theorem of the positivity indicators are showed:

Proposition 2 *If Pos is a positivity indicator on $\Im(\mathbb{R})$ then it holds:*

(i) $\underline{x} \geq 0 \Longrightarrow Pos([x]) = 1$ *and* $\overline{x} < 0 \Longrightarrow Pos([x]) = 0$

(ii) $([x] \neq 0) \& (\underline{x} = -\overline{x}) \Longrightarrow Pos([x]) = \frac{1}{2}$

(iii) $k < 0 \Longrightarrow Pos(k[x]) = Pos(-[x])$

(iv) $Pos([\underline{x}, 0]) = Pos([-1, 0]) \;\; \forall \underline{x} < 0$.

Theorem 3 *(Characterization) A map $Pos : \Im(\mathbb{R}) \longrightarrow [0,1]$ is a positivity indicator on $\Im(\mathbb{R})$ iff there exists a non-decreasing map $g : [0,1] \rightarrow [0, \frac{1}{2}]$ verifying $g(1) = \frac{1}{2}$ and* *related to Pos by:*

$$Pos([x]) = \begin{cases} 0 & if \quad \overline{x} < 0 \\ g(\frac{\overline{x}^+}{\underline{x}^-}) & if\ \underline{x} < 0\ and\ \overline{x}^+ \leqq \underline{x}^- \\ 1 - g(\frac{\underline{x}^-}{\overline{x}^+}) & if\ \underline{x} < 0\ and\ \overline{x}^+ > \underline{x}^- \\ 1 & if \quad \underline{x} \geq 0 \end{cases}$$

The non decreasing maps g in the characterization theorem can be used to easily determine positivity indicators. This is illustrated by means two examples.

First, the usual three valued logic characterizations of the assertions '$[x]$ is a positive interval', are recovered employing the maps $g_\perp$ and $g_\top$ defined in $[0,1]$ by:

$$g_\perp(t) = \begin{cases} 0\ if\ 0 \leqq t < 1 \\ \frac{1}{2}\ if \quad t = 1 \end{cases} \text{ and } g_\top(t) = \frac{1}{2}$$

the following positivity indicators $Pos_{g_\perp}$ and $Pos_{g_\top}$ on $\mathfrak{I}(\mathbb{R})$ are obtained:

$$Pos_{g_\perp}([x]) = \begin{cases} 0\ if & \overline{x}^+ < \underline{x}^- \\ \frac{1}{2}\ if & 0 \neq \overline{x}^+ = \underline{x}^- \\ 1\ if & [x] = 0 \text{ or } \overline{x}^+ > \underline{x}^- \end{cases} =$$

$$= \begin{cases} 0\ if & \frac{\underline{x}+\overline{x}}{2} < 0 \\ \frac{1}{2}\ if & [x] \neq 0 \text{ and } \frac{\underline{x}+\overline{x}}{2} = 0 \\ 1\ if & [x] = 0 \text{ or } \frac{\underline{x}+\overline{x}}{2} > 0 \end{cases}$$

$$Pos_{g_\top}([x]) = \begin{cases} 0\ if & \overline{x} < 0 \\ \frac{1}{2}\ if & \underline{x} < 0 \leqq \overline{x} \\ 1\ if & 0 \leqq \underline{x} \end{cases}$$

In Interval Analysis, $Pos_{g_\perp}$ and $Pos_{g_\top}$ represent three valued logic operators that evaluate the assertion '$[x]$ is positive'. From now on, additional $[0,1]$-valued characterizations type Pos, (between $Pos_{g_\perp}$ and $Pos_{g_\top}$), can be defined:

Example 4

Employing the maps $g_l(t) = \frac{t}{2}$ and $g_s(t) = \frac{t}{t+1}, t \in [0,1]$, the respectively associated indicators Pos_{g_l} and Pos_{g_s} are obtained:

$$Pos_{g_l}([x]) = \begin{cases} 0 & if \quad \overline{x} < 0 \\ -\frac{\overline{x}}{2\underline{x}} & if\ \underline{x} < 0, \underline{x} + \overline{x} \leqq 0 \\ 1 + \frac{\underline{x}}{2\overline{x}} & if\ \underline{x} < 0, \underline{x} + \overline{x} > 0 \\ 1 & if \quad \underline{x} \geq 0 \end{cases}$$

$$Pos_{g_s}([x]) = \begin{cases} 0 & if \quad \overline{x} < 0 \\ \frac{\overline{x}}{\overline{x} - \underline{x}} & if\ \underline{x} < 0, \overline{x} \geq 0 \\ 1 & if \quad \underline{x} \geq 0 \end{cases}$$

For example:

$Pos_{g_\perp}([-2,1]) = 0$, $Pos_{g_\perp}([-1,3]) = 1$, $Pos_{g_\perp}([-1,0]) = 0$

$Pos_{g_\top}([-2,1]) = Pos_{g_\top}([-1,3]) = Pos_{g_\top}([-1,0]) = \frac{1}{2}$

$Pos_{g_l}([-2,1]) = \frac{1}{4}$, $Pos_{g_l}([-1,3]) = \frac{5}{6}$, $Pos_{g_l}([-1,0]) = 0$

$Pos_{g_s}([-2,1]) = \frac{1}{3}$, $Pos_{g_s}([-1,3]) = \frac{3}{4}$, $Pos_{g_s}([-1,0]) = 0$.

Other fuzzy indicators on $\mathfrak{I}(\mathbb{R})$ are proposed:

Definition 5 *If Pos is a positivity indicator on $\mathfrak{I}(\mathbb{R})$ then the associated negativity, strictly positivity and strictly negativity indicators are defined by: $Neg([x]) = Pos(-[x])$, $SPos([x]) = 1 - Pos(-[x])$ and $SNeg([x]) = 1 - Pos([x])$ $\forall [x] \in \mathfrak{I}(\mathbb{R})$.*

3.2 Comparison indexes

Fuzzy comparison indexes on the set of interval numbers are defined:

Definition 6 *If Pos is a positivity indicator on $\mathfrak{I}(\mathbb{R})$ then the associated comparison index R_{Pos} in $\mathfrak{I}(\mathbb{R}) \times \mathfrak{I}(\mathbb{R})$ is defined by:*

$R_{Pos}([x],[y]) = Pos([y] - [x]) = Pos([\underline{y} - \overline{x}, \overline{y} - \underline{x}])$ $\forall([x],[y]) \in \mathfrak{I}(\mathbb{R}) \times \mathfrak{I}(\mathbb{R})$.

Some properties of the comparison indexes R_{Pos} are showed.

3.3 Extensions of Pos and R_{Pos} to the fuzzy quantities

The extensions of Pos and R_{Pos} to fuzzy intervals rest on the following notion of average index (see Campos and González [3]):

Definition 7 (Campos and González) *If Y denotes a subset in $[0,1]$, F denotes a probability distribution in Y and $\varphi_A : Y \longrightarrow \mathbb{R}$ is a map which represents the position of every α-cut of A in $\mathbb{R}$, then the number $V_F(A) = \int_Y \varphi_A(\alpha) dF(\alpha)$ is called average index of $A \in \mathcal{F}\mathfrak{I}(\mathbb{R})$.*

In this way, the following extensions are proposed.

Definition 8 *Let F be a distribution function in $\mathbb{R}$ such that $F(0) = 0$ and $F(1) = 1$. Then the F-extensions of the positivity indicator Pos and the comparison index R_{Pos} to fuzzy quantities A, B in $\mathcal{F}\mathfrak{I}(\mathbb{R})$ are defined (if exist) by:*

$$Pos_F(A) = \int_{0+}^{1} Pos([\underline{a_\alpha}, \overline{a_\alpha}]) dF(\alpha) \ ,$$

$$R_{Pos_F}(A,B) = \int_{0+}^{1} R_{Pos}([\underline{a_\alpha}, \overline{a_\alpha}], [\underline{b_\alpha}, \overline{b_\alpha}]) dF(\alpha).$$

Properties for the indicators Pos_F are showed.

Proposition 9 *The indicators type Pos_F are fuzzy subsets in $\mathcal{F}\mathfrak{I}(\mathbb{R})$ verifying:*

(P1) $Pos_F(x) = 1 \ \ if \ \ x \geq 0$ and $Pos_F(x) = 0 \ \ if \ \ x < 0$.

And, supposing the existence of $Pos_F(A)$ and $Pos_F(B)$:

(P2) $A \preccurlyeq B \implies Pos_F(A) \leqq Pos_F(B)$.

(P3) $Pos_F(kA) = Pos_F(A) \, \forall k > 0 \, \forall A \in \mathcal{F}\mathfrak{I}(\mathbb{R})$.

(P4') If A is a continuous map in $x = 0$, then $Pos_F(-A) = 1 - Pos_F(A)$.

Conclusions. We have presented ordering procedures of interval numbers and of fuzzy quantities in the frame of the fuzzy relations. First, we have considered fuzzy interpretations $Pos \in [0,1]^{\mathfrak{I}(\mathbb{R})}$ of the assertion '$[x]$ is a positive interval'. We have characterized every fuzzy subset type Pos by a non-decreasing real map $g : [0,1] \to [0, \frac{1}{2}]$. Finally, comparison indexes between interval numbers and fuzzy quantities are obtained via the maps type Pos, the interval arithmetic and the average indexes.

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