On Fuzzy Inference by the Least Squares Method*

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Abstract

A very different suggestion to the Zadeh's rule for fuzzy inference is the least squares method proposed by T. D. Pham and Valliappan. The aim of this paper is to study the possible contradictions appearing in this method and, additionally, to extend it when functions are not necessarily monotonic.

Keywords: Generalized modus ponens, fuzzy inference rules, least squares model.

1 Introduction

1.1 Inference is a process by means of which new knowledge can be attained from known facts (premises). In classical logic, where only two values can give the truthfulness of a sentence, the method commonly used to make an inference is Modus Ponens (MP). As is well known, the scheme of this method can be formulated as:

$$\begin{array}{lll} P_1: & \text{If } A & \text{then} \quad B \\ P_2: & A & \\ \hline C: & & B \end{array}$$

where P_1 and P_2 are the premises and C is the ultimate conclusion of the method. This scheme is extended to fuzzy reasoning as what is known as Generalized Modus Ponens (GMP), which is described as follows:

$$\begin{array}{llll} P_1: & \text{If } x \text{ is } A & \text{then} & y \text{ is } B \\ P_2: & x \text{ is } A^* & & \\ \hline C: & & & y \text{ is } B^* \end{array} \qquad (1)$$

where A and A^* are predicates in a universe of discourse E, whose membership functions μ_A, μ_{A^*} are fuzzy subsets of E, that is, $\mu_A, \mu_{A^*} \in [0,1]^E$, and B, B^* are predicates in another universe F, with membership functions $\mu_B, \mu_{B^*} \in [0,1]^F$.

It should be pointed out that if the predicates are crisp and $A = A^*$, then GMP coincides with classical MP.

One of the best known methods for obtaining the above rule of fuzzy inference was developed by L. Zadeh (see [6], [7] and [8]). This is the well-known Compositional Inference Rule, where the conclusion B^* is obtained as follows: For all $y \in F$

$$\mu_{B^*}(y) = \sup_{x \in E} T(\mu_{A^*}(x), \mu_R(x,y)) \qquad (1)$$

with $\mu_R(x,y) = J(\mu_A(x), \mu_B(y))$ is a fuzzy relation, where usually J is an implication function, modelling the rule "If-Then" in accordance with its use, and T is a t-norm translating the conjunction of the premises P_1 and P_2.

1.2 A very different suggestion to Zadeh's for fuzzy inference was proposed by T.D. Pham and S. Valliappan in [3] for a finite universe. The general idea is to link μ_A and μ_{A^*} by constants, making it possible to express μ_{A^*} by μ_A. Having done this, this relation is translated to μ_B to obtain μ_{B^*}.

μ_A and μ_{A^*} are related not directly, but by other functions that approximate them, and that are searched by the least squares method. The process is as follows: Let $E = \{x_1, \ldots, x_n\}$ be a finite universe of discourse and $\mu_A, \mu_{A^*} \in [0,1]^E$ be two fuzzy subsets of E. Two fuzzy sets $\mu_{\tilde{A}}, \mu_{\tilde{A}^*} \in [0,1]^E$ are obtained, defining $\mu_{\tilde{A}}(x) = ax^b$ and $\mu_{\tilde{A}^*}(x) = a^* x^{b^*}$, for all $x \in E$, where the constants a, b, a^*, b^* can be determined by means of the least squares method as follows: the square distance

$$\Phi(a,b) = \|(\ln(\mu_A(x_1)), .., \ln(\mu_A(x_n))) - (\ln(ax_1^b), .., \ln(ax_n^b))\|^2$$

is minimized in relation to the variables a and b, thus making $\frac{\partial \Phi}{\partial a}(a,b) = 0$ and $\frac{\partial \Phi}{\partial b}(a,b) = 0$, the linear alge-

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braic system

$$n \ln a + \left(\sum_{i=1}^{n} \ln x_i \right) b = \sum_{i=1}^{n} \ln(\mu(x_i))$$
$$\left(\sum_{i=1}^{n} \ln x_i \right) \ln a + \left(\sum_{i=1}^{n} (\ln x_i)^2 \right) b = \sum_{i=1}^{n} (\ln x_i)(\ln(\mu(x_i)))$$

is obtained; and the same goes for a^*, b^*. Once a, b, a^*, b^* are found, for all $x \in E$ it follows

$$\mu_{\tilde{A}^*}(x) = c_1 \mu_{\tilde{A}}(x)^{c_2}$$

where $c_1 = \frac{a^*}{a^{b^*/b}}$ and $c_2 = \frac{b^*}{b}$. So, taking into account that $\mu_{\tilde{A}}$ and $\mu_{\tilde{A}^*}$ are good approximations of μ_A and μ_{A^*}, respectively, we can conclude that

$$\mu_{A^*} \approx c_1 \mu_A^{c_2},$$

relation that will be translated to infer conclusions, that is, μ_{B^*} will be defined for all $x \in E$ as

$$\mu_{B^*}(x) = c_1 \mu_B(x)^{c_2}.$$

Note that a restriction using this method is that the universe of discourse E must be finite. On the other hand, it can be that $\mu_{\tilde{A}}(x) \notin [0,1]$ or $\mu_{\tilde{A}^*}(x) \notin [0,1]$ for some $x \in E$. However, some conditions on the constants a, b, a^*, b^* are given in [1] to ensure that $\mu_{\tilde{A}}, \mu_{\tilde{A}^*} \in [0,1]^E$. Nevertheless, this fact is not relevant for the inference, as the aim is to capture the relationship between μ_A, μ_{A^*} and $\mu_{\tilde{A}}, \mu_{\tilde{A}^*}$.

On the other hand, the contradictions in the Compositional Rule of Inference were studied in [5], giving some conditions to ensure that contradictory outputs are not obtained from given inputs. We can conduct a similar study with the method of inference we are working on.

This paper is devoted to:

1. Studying the possible contradictions appearing in the least squares model for fuzzy rules of inference, and

2. extending this method when the functions μ_A and μ_{A^*} are more general and not necessarily monotonic.

2 Contradiction and the Least Squares Method

In any inference of the type described by scheme (1) no contradiction should appear in the sense that:

1. A, A^*, B and B^* must not be self-contradictory

2. A and A^* must not be contradictory, and

3. B and B^* must not be contradictory.

So, firstly, we must clarify when a fuzzy set is self-contradictory and when two fuzzy sets are contradictory.

Given a strong negation N (see [4]), $\mu \in [0,1]^E$ is N-contradictory with $\sigma \in [0,1]^E$, if $\mu \leq N \circ \sigma$. $\mu \in [0,1]^E$ is contradictory with $\sigma \in [0,1]^E$, if a negation N exists such that $\mu \leq N \circ \sigma$. Furthermore, μ is self-contradictory, if $\mu \leq N \circ \mu$ for some N. [5] gives some conditions for contradiction. In particular:

Theorem 2.1. $\mu \in [0,1]^E$ is self-contradictory if and only if $\operatorname{Sup}_{x \in E} \mu(x) \neq 1$

In the case we are working on, proposed by [1], the universes are finite and the above result is equivalent to:

Theorem 2.2. $\mu \in [0,1]^E$ is self-contradictory if and only if $\operatorname{Max}_{x \in E} \mu(x) \neq 1$

Furthermore, [5] provides some necessary conditions for sets μ and σ to be contradictory. Moreover, as we are working on finite universes, some necessary and sufficient conditions, are obtained in the following theorem:

Theorem 2.3. Let μ and σ be fuzzy sets in a finite universe E. μ and σ are contradictory if and only if, for all $x \in E$ if $\mu(x) = 1$, then $\sigma(x) = 0$, and if $\sigma(x) = 1$, then $\mu(x) = 0$.

Proof: If μ and σ are contradictory, a strong negation N exists such that $\mu(x) \leq N \circ \sigma(x)$ for all $x \in E$. If, for some $x_0 \in E$, $\mu(x_0) = 1$, then $N \circ \sigma(x_0) = 1$, and, hence, $\sigma(x_0) = 0$. Similarly, if $\sigma(x_0) = 1$, then $\mu(x_0) = 0$.

Let us see the reciprocal. Let $h = \max\{\sigma(x) \,;\, \mu(x) > 0\}$. Note that $h < 1$, as if $\sigma(x_0) = 1$ with $\mu(x_0) > 0$, the hypothesis does not hold. And let $k = \max\{\mu(x) \,;\, \mu(x) < 1\}$.

Let us choose m, such that $\max(h, k) < m < 1$, and let N_m be the negation with fixed point m. Let us prove that μ and σ are N_m-contradictory, and, hence contradictory, that is, $\sigma(x) \leq N_m \circ \mu(x)$ for all $x \in E$.

- If $\mu(x) = 0$, $N_m \circ \mu(x) = 1$, and $\sigma(x) \leq N_m \circ \mu(x)$.
- If $\mu(x) = 1$, by hypothesis, $\sigma(x) = 0$, and $\sigma(x) \leq N_m \circ \mu(x)$.
- Finally, if $0 < \mu(x) < 1$, $\sigma(x) < m$ as $\mu(x) > 0$ and, on the other hand, as $\mu(x) < 1$, $\mu(x) \leq k < m$, then $N_m(m) = m \leq N_m(\mu(x))$ and, therefore, $\sigma(x) < m \leq N_m \circ \mu(x)$. □

Let us apply these results to our case.

The first step for ruling out contradictions is to ensure that the predicates A, A^* and B are not self-contradictory; that is, there must be some $x \in E$ such that $\mu_A(x) = 1$ (and similarly for A^* and B).

The second step is to ensure that B^* is not self-contradictory, that is, there must be $x \in E$ such that $\mu_{B^*}(x) = c_1 \mu_B^{c_2}(x) = 1$, that holds if and only if $\mu_B(x) = \frac{1}{c_1^{1/c_2}}$, but it could be that no element verifies this condition. A possible solution to avoid contradiction in this case is to determine $\mu_{B^*}(x) = 1$ for all x such that $\mu_B(x) = 1$, and $\mu_{B^*}(x) = c_1 \mu_B^{c_2}(x)$ for the remainder.

Furthermore, A and A^* must not be contradictory. However, it is sufficient that a $x \in E$ exists such that $\mu_A(x) = 1$ and $\mu_{A^*}(x) > 0$ or viceversa.

The last condition is not to obtain a consequence that is contradictory to the consequent of the If-Then sentence, that is, μ_B and μ_{B^*} must not be contradictory. But, with the previous modification made to the fuzzy set μ_{B^*}, this fuzzy set is, evidently, not contradictory with μ_B.

3 Fuzzy Inference using Least Squares for Non-monotonic Functions

In this section, we focus on how to use the least squares method to make inferences when the membership functions are not monotonic.

Let $E = \{x_1, \ldots, x_n\}$ and $F = \{z_1, \ldots, z_m\}$ be two universes of discourse in $[0,1]$, A, A^* predicates in E and B a predicate in F whose membership functions are $\mu_A, \mu_{A^*} \in [0,1]^E$ and $\mu_B \in [0,1]^F$, respectively. The aim is to find B^* verifying the above scheme (1). The method will be developed stepwise:

Step one

Given μ_A, μ_{A^*} fuzzy sets in E, let us consider $\{y_1, \ldots, y_k, y_{k+1}\} \subset E$ verifying that for every $i \in \{1, \ldots, k\}$ both $\mu_A|_{[y_i, y_{i+i}] \cap E}$ and $\mu_{A^*}|_{[y_i, y_{i+i}] \cap E}$ are monotonic functions. For the sake of simplification, we will denote $\mu_A|_{[y_i, y_{i+1}] \cap E} = \mu_{A_i}$ and $\mu_{A^*}|_{[y_i, y_{i+1}] \cap E} = \mu_{A_i^*}$.

Step two

As functions μ_{A_i} and $\mu_{A_i^*}$ are monotonic, they can be approximated using the least squares method (as in [1] and [3]) by $a_i x^{b_i}$ and $a_i^* x^{b_i^*}$, respectively, obtaining the relation $\mu_{A_i^*} \approx c_1^i \mu_{A_i}^{c_2^i}$ for each $i \in \{1, \ldots, k\}$.

Step three

Translating the above relation to obtain consequences, for each $i \in \{1, \ldots, k\}$, we define

$$\mu_{B_i^*}(x) = c_1^i \mu_B(x)^{c_2^i}, \quad \forall x \in F$$

Step four

Finally, to obtain a consequence B^* according to scheme (1), the function

$$\mu_{B^*}(x) = F(\mu_{B_1^*}(x), \ldots, \mu_{B_k^*}(x)), \ \forall x \in F$$

will be obtained, where F is an aggregation function (see [2]). It would be advisable in the choice of F to use operators that ensure a result between the maximum and the minimum of these functions, that is, averaging operators, Min $\leq F \leq$ Max.

Example

Let $E{=}F{=}\{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$ be a universe of discourse. Let A = "close to 0.5" and B= "big" be predicates in E whose compatibility functions are

$$\mu_A = 0|0 + 0|0.1 + 0.25|0.2 + 0.5|0.3 + 0.75|0.4 + 1|0.5 + 0.75|0.6 + 0.5|0.7 + 0.25|0.8 + 0|0.9 + 0|1$$

and $\mu_B = id|_E$. Let us suppose that the rule

"If x is A, then y is B"

is verified, and the given information is A^*= "very close to 0.5", whose compatibility function is

$$\mu_{A^*} = 0|0 + 0|0.1 + 0.0625|0.2 + 0.25|0.3 + 0.5625|0.4 + 1|0.5 + 0.4219|0.6 + 0.125|0.7 + 0.0156|0.8 + 0|0.9 + 0|1$$

We will find B^*.

In the intervals $[0, 0.5] \cap E = E_1$ and $[0.5, 1] \cap E = E_2$ both functions μ_A and μ_{A^*} are monotonic, and they verify

$$\mu_{A_1^*} = \mu_{A^*}|_{E_1} = (\mu_A|_{E_1})^2 = \mu_{A_1}^2$$
$$\mu_{A_2^*} = \mu_{A^*}|_{E_2} = (\mu_A|_{E_2})^3 = \mu_{A_2}^3$$

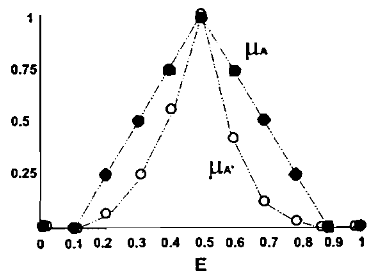

Figure 1: Representation of A="close to 0.5" and A^*="very close to 0.5".

It was proved in [1] that if $\mu_{A^*} = \mu_A^p$, $p \geq 0$, then the power functions approximating them using the least squares method, $\mu_{\tilde{A}}$ and $\mu_{\tilde{A}^*}$, verify the same relation $\mu_{\tilde{A}} = \mu^p_{\tilde{A}^*}$. So, translating this relation to the consequences, $B_1^* = B^2$ and $B_2^* = B^3$, that is:

$$\mu_{B_1^*} = 0|0 + 0.01|0.1 + 0.04|0.2 + 0.09|0.3 + 0.16|0.4 + 0.25|0.5 + 0.36|0.6 + 0.49|0.7 + 0.64|0.8 + 0.81|0.9 + 1|1$$

and

$$\mu_{B_2^*} = 0|0 + 0.001|0.1 + 0.008|0.2 + 0.027|0.3 + 0.064|0.4 + 0.125|0.5 + 0.216|0.6 + 0.343|0.7 + 0.512|0.8 + 0.729|0.9 + 1|1$$

Finally, B^* is obtained:

$$\mu_{B^*}(x) = F(\mu_{B_1^*}(x), \mu_{B_2^*}(x)), \ \forall x \in E$$

where $F : [0,1]\times[0,1] \to [0,1]$ is an averaging operator. In particular, let us consider the following examples:

- $F = \text{Max}$, then $\mu_{B^*} = \mu_B^2$.
- F is a weighted mean that can be chosen according to the weight of each function $\mu_{\tilde{A}_i}$, $i = 1, 2$, in the whole universe. In this case, it could be $F(x, y) = \dfrac{x+y}{2}$, obtaining

$$\mu_{B^*} = 0|0 + 0.0055|0.1 + 0.024|0.2 + 0.0585|0.3 + 0.112|0.4 + 0.1875|0.5 + 0.288|0.6 + 0.4165|0.7 + 0.576|0.8 + 0.7695|0.9 + 1|1$$

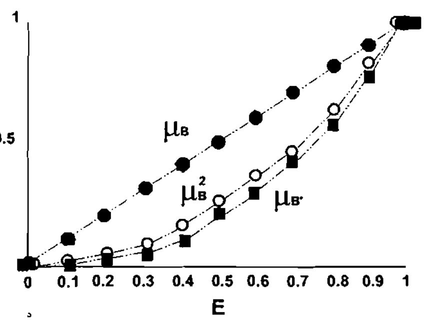

Figure 2: Representation of B="big", B^2 and B^*.

Remarks

1. Before applying the proposed process, the conditions obtained in the section 2 must be taken into account in order to avoid contradictions in this inference.

2. Note that the method does not guarantee that the function μ_{B^*} is a fuzzy set, as it can take values not belonging to $[0, 1]$. Some conditions on when this holds were given in [1].

3. If $\mu_{A^*} = \mu_A^p$, with $p \geq 0$, then $\mu_{B^*} = F(\mu_{B_1^*}, \ldots, \mu_{B_k^*}) = F(\mu_B^p, \ldots, \mu_B^p) = \mu_B^p$. So, if A^*="very A" then B^*="very B". This fact can suitably translate the reality in some frameworks, but in general, the Zadeh's compositional rule of inference does not capture it. For example, in $E = F = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$, let $\mu_A(x) = x$ and $\mu_b(y) = 1 - y$ be the membership functions of the predicates A =big and B =small. Taking the rule "If x is big then y is small" which can be represented by the Kleen-Dienes' implication, $J(x, y) = \text{Max}(1 - x, y)$, and the Lukasiewicz t-norm $T = W$ then, using the Zathe's rule for the observation A^* =very big, the conclusion is

$$\mu_{B^*}(y) = \sup_{x \in E} W(x^2, \text{Max}(1-x, 1-y))) = 1 - y = \mu_B(y)$$

4. It would be desirable for any fuzzy inference to capture the classical case in the sense that if $\mu_{A^*} = \mu_A$, then $\mu_{B^*} = \mu_B$; taking in account the previous note with $p = 1$, this fact is verified in the inference by the least squares method.

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