The Construction of Compensation Aggregation Operators

Content Areas: Reasoning under uncertainty
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Abstract

The issue of constructing compensation operators is discussed in this paper. We propose three construction methods, and construct a series of compensation operators by using these methods.

1 Introduction

Many researchers define the aggregation as operators generalizing "AND" and "OR" fuzzy connectives [Portilla *et al.*, 2000]. However, the two extremal situations of "AND" and "OR" may not be able to match real-life scenario. Thus, other alternative aggregation operators, such as mean operators [Aczél and Saaty, 1983], OWA-operators [Yager, 1988] and compensation operators [Yager *et al.*, 1996; Klement *et al.*, 1996], have been proposed for a tradeoff between these two cases.

Compensation operators are applicable in many real problems, *e.g.*, automated negotiation problems [Luo *et al.*, 2000b] in e-commerce, meeting scheduling problems [Luo *et al.*, 2000a], solution synthesis [M. Zhang and C. Zhang, 1999] in distributed expert systems, and parallel combination [Luo and Zhang, 1999] of uncertainties in expert systems. The best choice of compensation operators may vary from problem to problem. In order to offer more freedom in the selection of suitable compensation operators for various specific applications, the paper discusses the issue of constructing compensation operators.

The rest of this paper is organized as follows. Section 2 gives three methods for constructing compensation operators. Sections 3-5 uses these methods to construct a series of compensation operators. The last section concludes the paper.

2 Basic Principle

This section will give three methods for constructing compensation operators.

First, let us recall the concept of compensation operators.

Definition 1 ([Yager *et al.*, 1996]) *A binary operator* $\boxplus^{(\varepsilon)}$: $[0,1] \times [0,1] \to [0,1]$ *is called a* uninorm operator *if it is increasing, associative and commutative and has unit element* $\varepsilon \in [0,1]$. *In particular, when* ε *is* 1, 0, *and between* 1 *and* 0, *respectively, a uninorm operator with unit element* ε *is called T-norm, T-conorm, and compensation operator, respectively.*

In a multi-agent environment, suppose each agent evaluates the same object. Intuitively, in the case their evaluations are positive, the evaluations should enhance each other; in the case their evaluations are negative, the evaluations should weaken each other; in the case some evaluations are positive and others are negative, there should be a tradeoff. When the evaluations take values on $[0,1]$, using a compensation operator to aggregate the evaluations can capture the intuition [Klement *et al.*, 1996]. In fact, we just need to regard the unit element of a compensation operator as a threshold: if an evaluation is greater that the threshold the evaluation is regarded as being positive; otherwise, the evaluation is regarded as being negative.

Second, let us develop the first method.

Definition 2 *An operator* $\circ : [-1,1] \times [-1,1] \longrightarrow [-1,1]$ *is called a T-conorm-like operator, denoted as* $\bigtriangledown'$, *if is increasing, associative and commutative, and has unit element* 0.

The following lemma is a basic fact in modern algebra [Marcus, 1978].

Lemma 1 *Let* $(\circ, X)$ *and* $(\odot, Y)$ *be two algebraic structures. If*

$$y_1 \odot y_1 = f^{-1}(f(y_1) \circ f(y_2)),$$

where mapping $f : X \to Y$ *is 1-1 and increasing, and operator* $\circ$ *is increasing, associative and commutative, then operator* $\odot$ *is increasing, associative and commutative.*

Theorem 1 *The following operator is a compensation operator:*

$$a_1 \boxplus^{(\varepsilon)}_{\bigtriangledown'} a_2 = h^{-1}(h(a_1) \bigtriangledown' h(a_2)), \quad (1)$$

where $h : [0,1] \to [-1,1]$ *is an 1-1 increasing map satisfying* $h(0) = -1$, $h(1) = 1$, *and* $h(\varepsilon) = 0$.

Proof. By Lemma 1, commutativity, associativity and monotonicity hold for operator $\boxplus^{(\varepsilon)}_{\bigtriangledown'}$. And its unit element is ε. In fact, $\forall a \in [0,1]$, we have

$$a \boxplus^{(\varepsilon)}_{\bigtriangledown'} \varepsilon = h^{-1}(h(a) \bigtriangledown' h(\varepsilon)) = h^{-1}(h(a) \bigtriangledown' 0) = h^{-1}(h(a)) = a.$$

□

Third, let us develop the second method.

Lemma 2 ([Klement *et al.*, 1996]) *A operator* $\oplus_P : [0,1] \times [0,1] \to [0,1]$*, defined as*

$$a_1 \oplus_P a_2 = \frac{a_1 a_2 (1-\lambda)}{(1-a_1)(1-a_2)\lambda + a_1 a_2 (1-\lambda)}, \quad (2)$$

where $\lambda \in (0,1)$*, is a uninorm operator with unit element* λ.

Actually, (2) is the parallel combination formula in the PROSPECTOR uncertain reasoning model [Duda *et al.*, 1976; Luo *et al.*, 1999].

Definition 3 *An 1-1 increasing function* $h : [-1,1] \to [0,1]$ *is said to be a generator function if* $h(-1) = 0$*,* $h(1) = 1$*, and* $h(0) = \lambda$.

Theorem 2 *The following is a compensation operator:*

$$a_1 \boxplus_{\oplus_P}^{(0.5)} a_2 = \frac{h^{-1}(h(2a_1-1) \oplus_P h(2a_2-1)) + 1}{2}, \quad (3)$$

where h is a generator function.

Proof. Let $g(x) = 2x - 1$, and thus $g^{-1}(x) = \frac{x+1}{2}$. And let $f(x) = h(g(x))$, and thus $f^{-1}(x) = g^{-1}(h^{-1}(x))$. Then we can rewrite (3) as

$$a_1 \boxplus_{\oplus_P}^{(0.5)} a_2 = f^{-1}(f(a_1) \oplus_P f(a_2)). \quad (4)$$

So, by Lemmas 1 and 2, commutativity, associativity and monotonicity holds for $\boxplus_{\oplus_P}^{(0.5)}$.

1) Unit element. $\forall a \in [0,1]$, we have

$$\begin{aligned} a \boxplus 0.5 &= f^{-1}(f(a) \oplus_P f(0.5)) = f^{-1}(f(a) \oplus h(g(0.5))) \\ &= f^{-1}(f(a) \oplus_P h(2 \times 0.5 - 1)) = f^{-1}(f(a) \oplus_P h(0)) \\ &= f^{-1}(f(a) \oplus_P \lambda) = f^{-1}(f(a)) = a. \end{aligned}$$

2) $\boxplus_{\oplus_P}$ is closed on $[0,1]$.

$$\begin{aligned} & 0 \le a_1 \le 1 \wedge 0 \le a_2 \le 1 \\ \Rightarrow\ & -1 \le 2a_1 - 1 \le 1 \wedge -1 \le 2a_2 - 1 \le 1 \\ \Rightarrow\ & 0 \le h(2a_1 - 1) \le 1 \wedge 0 \le h(2a_2 - 1) \le 1 \\ \Rightarrow\ & 0 \le h(2a_1 - 1) \le 1 \wedge 0 \le h(2a_2 - 1) \le 1 \\ \Rightarrow\ & 0 \le h(2a_1 - 1) \oplus_P h(2a_2 - 1) \le 1 \\ \Rightarrow\ & -1 \le h^{-1}(h(2a_1 - 1) \oplus_P h(2a_2 - 1)) \le 1 \\ \Rightarrow\ & 0 \le \frac{h^{-1}(h(2a_1 - 1) \oplus_P h(2a_2 - 1)) + 1}{2} \le 1 \\ \Rightarrow\ & 0 \le 2a_1 \boxplus_{\oplus_P} a_2 \le 1. \end{aligned}$$

□

Finally, the third method is:

Theorem 3 *If an 1-1 mapping* $h : [0,1] \to [0,1]$ *is increasing and satisfies* $h(0) = 0$*,* $h(1) = 1$ *and for two constant* $\varepsilon, \varepsilon' \in (0,1)$*,* $h(\varepsilon) = \varepsilon'$*, and* $\boxplus^{(\varepsilon')}$ *is a compensation operator, then the following is a compensation operator:*

$$a_1 \boxplus^{(\varepsilon)} a_2 = h^{-1}(h(a_1) \boxplus^{(\varepsilon')} h(a_2)). \quad (5)$$

Proof. By Lemma 1, $\boxplus^{(\varepsilon)}$ satisfies commutativity, associativity and monotonicity. Besides, ε is the unit element of $\boxplus^{(\varepsilon)}$. In fact, we have

$$\begin{aligned} a \boxplus^{(\varepsilon)} \varepsilon &= h^{-1}(h(a) \boxplus^{(\varepsilon')} h(\varepsilon)) = h^{-1}(h(a) \boxplus^{(\varepsilon')} \varepsilon') \\ &= h^{-1}(h(a)) = a. \end{aligned}$$

Therefore, $\boxplus^{(\varepsilon)}$ is a compensation operator. □

3 From T-Conorm-Like Operators

Sometimes a T-conorm on $[0,1]$ can become a T-conorm-like operator $[-1,1]$. In this section, we examine:

1) Boundary T-conorm: $a_1 \oplus a_2 = \min\{1, a_1 + a_2\}$, (6)

2) Zadeh T-conorm: $a_1 \vee a_2 = \max\{a_1, a_2\}$. (7)

Now we use boundary T-conorm operator (6) to construct a compensation operator. In order to guarantee the operator closed on $[-1,1]$, we change its definition to

$$a_1 \oplus' a_2 = \max\{-1, \min\{1, a_1 + a_2\}\}. \quad (8)$$

Corollary 1 *The following is a compensation operator:*

$$a_1 \boxplus_{\oplus} a_2 = \max\{-0.5, \min\{0.5, a_1 + a_2 - 1\}\} + 0.5. \quad (9)$$

Proof. The boundary T-conorm operator $\oplus'$ defined as (6) is a T-conorm-like operator on $[-1,1]$ because: 1) It is obvious that operator $\oplus'$ defined as (8) is closed on $[-1,1]$. 2) Commutativity. $a_1 \oplus a_2 = \min\{1, a_1 + a_2\} = \min\{1, a_2 + a_1\} = a_2 \oplus a_1$. 3) Associativity. $(a_1 \oplus a_2) \oplus a_3 = \min\{1, \min\{1, a_1 + a_2\} + a_3\} = \min\{1, a_2 + a_1 + a_3\} = \min\{1, a_1 + \min\{1, a_2 + a_3\}\} = a_1 \oplus (a_2 \oplus a_3)$. 4) Monotonicity. $a_1 \le a_1' \wedge a_2 \le a_2' \Rightarrow \min\{1, a_1 + a_2\} \le \min\{1, a_1' + a_2'\} \Rightarrow a_1 \oplus a_2 \le a_1' \oplus a_2'$. 5) Unit element. $a_1 \oplus 0 = \min\{1, a_1 + 0\} = a_1$.

Thus, by Theorem 1, from $h(x) = 2x - 1$ and operator $\oplus'$ defined as (8), we can obtain the following compensation

$$\begin{aligned} a_1 \boxplus_{\oplus} a_2 &= h^{-1}(\max\{-1, \min\{1, h(a_1) + h(a_2)\}\}) \\ &= \frac{\max\{-1, \min\{1, 2a_1 - 1 + 2a_2 - 1\}\} + 1}{2} \\ &= \max\{-0.5, \min\{0.5, a_1 + a_2 - 1\}\} + 0.5. \end{aligned}$$

□

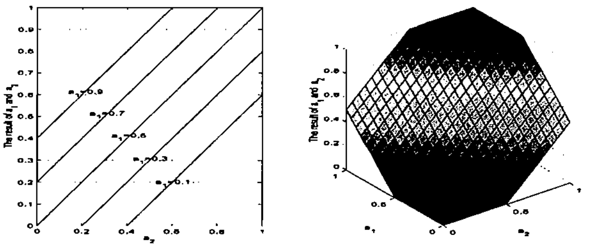


Figure 1. Contour plots of compensation operator $\boxplus_{\oplus'}$.

Notice that we cannot always use a T-conorm operator to construct a compensation operator. For example, we cannot construct a compensation operator from Zadeh T-conorm operator (7). This is because the operator cannot become a T-conorm-like operator on $[-1,1]$. In fact, 0 is not the unit element of Zadeh T-conorm operator *max* on $[-1,1]$.

4 From Operator $\oplus_P$

The section constructs compensation operators from $\oplus_P$.

Corollary 2 *The following is a compensation operator:*

$$a_1 \boxplus_{CF} a_2 = \begin{cases} 2(a_1 + a_2 - a_1 a_2) - 1 & \text{if } a_1 > 0.5, a_2 > 0.5, \\ 2a_1 a_2 & \text{if } a_1 < 0.5, a_2 < 0.5, \\ \frac{a_1 + a_2 - 1}{1 - min(|2a_1 - 1|, |2a_2 - 1|)} + 0.5 & \text{if } a_1 \le 0.5 \le a_2 \text{ or } a_2 \le 0.5 \le a_1. \end{cases} \quad (10)$$

Proof. We use Theorem 2 to prove the corollary. First, function $h : [-1,1] \to [0,1]$, defined as follows, is a generator function:

$$h(x) = \begin{cases} \frac{\lambda}{1-(1-\lambda)x} & \text{if } x \geq 0, \\ \frac{(x+1)\lambda}{1+\lambda x} & \text{if } x < 0, \end{cases} \tag{11}$$

where $\lambda \in (0,1)$. In fact, clearly $h(-1) = 0$, $h(0) = \lambda$ and $h(1) = 1$, and

$$\begin{aligned}
&0 < a_1 \leq a_2 \Rightarrow 1-(1-\lambda)a_1 \geq 1-(1-\lambda)a_2 \\
&\Rightarrow \frac{\lambda}{1-(1-\lambda)a_1} \leq \frac{\lambda}{1-(1-\lambda)a_2} \Rightarrow h(a_1) \leq h(a_2), \\
&a_1 \leq a_2 \leq 0 \Rightarrow (1-\lambda)a_1 \leq (1-\lambda)a_2 \\
&\Rightarrow 1 + a_1a_2\lambda + a_1 - \lambda a_1 \leq 1 + a_1a_2\lambda + a_2 - \lambda a_2 \\
&\Rightarrow 1 + a_1 + a_1a_2\lambda + \lambda a_2 \leq 1 + a_2 + a_1a_2\lambda + \lambda a_1 \\
&\Rightarrow (1+a_1)(1+\lambda a_2) \leq (1+a_2)(1+\lambda a_1) \\
&\Rightarrow \frac{(1+a_1)\lambda}{1+\lambda a_1} \leq \frac{(1+a_2)\lambda}{1+\lambda a_2} \Rightarrow h(a_1) \leq h(a_2), \\
&-1 \leq a_1 \leq 0 \leq a_2 \Rightarrow (1+a_1)a_2 \geq 0 \geq a_1 \\
&\Rightarrow (1+a_1)(1-\lambda)a_2 \geq (1-\lambda)a_1 \\
&\Rightarrow \lambda a_1 \geq a_1 - (a_1+1)(1-\lambda)a_2 \\
&\Rightarrow 1+\lambda a_1 \geq a_1 + 1 - (a_1+1)(1-\lambda)a_2 \\
&\Rightarrow \frac{1}{1+(1-\lambda)a_2} \geq \frac{a_1+1}{1+\lambda a_1} \\
&\Rightarrow \frac{\lambda}{1+(1-\lambda)a_2} \geq \frac{(a_1+1)\lambda}{1+\lambda a_1} \Rightarrow h(a_1) \leq h(a_2).
\end{aligned}$$

Therefore, according to Theorem 2, by using h and

$$h^{-1}(x) = \begin{cases} \frac{x-\lambda}{x(1-\lambda)} & \text{if } x > \lambda, \\ \frac{x-\lambda}{\lambda(1-x)} & \text{if } x \leq \lambda, \end{cases}$$

we can construct the following compensation operator:

1) $\forall a_1, a_2 \in (0.5, 1]$, we have

$$\begin{aligned}
&a_1 \boxplus_{CF}^{(0.5)} a_2 \\
&= \frac{h^{-1}\left(\frac{1}{\frac{(1-h(2a_1-1))(1-h(2a_2-1))\lambda}{h(2a_1-1)h(2a_2-1)(1-\lambda)}+1}\right)+1}{2} \\
&= \frac{h^{-1}\left(\frac{1}{\frac{\left(1-\frac{\lambda}{1-(1-\lambda)(2a_1-1)}\right)\left(1-\frac{\lambda}{1-(1-\lambda)(2a_2-1)}\right)\lambda}{\frac{\lambda}{1-(1-\lambda)(2a_1-1)}\times\frac{\lambda}{1-(1-\lambda)(2a_2-1)}\times(1-\lambda)}+1}\right)+1}{2} \\
&= \frac{h^{-1}\left(\frac{\lambda}{4(1-a_1)(1-a_2)(1-\lambda)+\lambda}\right)+1}{2}.
\end{aligned}$$

Notice that we have

$$\begin{aligned}
&\frac{\lambda}{4(1-a_1)(1-a_2)(1-\lambda)+\lambda} - \lambda \\
&= \lambda(1-\lambda)\frac{1-4(1-a_1)(1-a_2)}{4(1-a_1)(1-a_2)(1-\lambda)+\lambda} \\
&\geq \lambda(1-\lambda) \times \frac{1-4(1-0.5)(1-0.5)}{4(1-a_1)(1-a_2)(1-\lambda)+\lambda} = 0.
\end{aligned}$$

Thus, further we have

$$\begin{aligned}
a_1 \boxplus_{CF}^{(0.5)} a_2 &= \frac{h^{-1}\left(\frac{\lambda}{4(1-a_1)(1-a_2)(1-\lambda)+\lambda}\right)+1}{2} \\
&= \frac{\frac{\frac{\lambda}{4(1-a_1)(1-a_2)(1-\lambda)+\lambda}-\lambda}{\frac{\lambda}{4(1-a_1)(1-a_2)(1-\lambda)+\lambda}\times(1-\lambda)}+1}{2} \\
&= 2(a_1+a_2-a_1a_2)-1.
\end{aligned}$$

2) $\forall a_1, a_2 \in [0, 0.5)$, we have

$$\begin{aligned}
&a_1 \boxplus_{CF}^{(0.5)} a_2 \\
&= \frac{h^{-1}\left(\frac{1}{\frac{(1-h(2a_1-1))(1-h(2a_2-1))\lambda}{h(2a_1-1)h(2a_2-1)(1-\lambda)}+1}\right)+1}{2} \\
&= \frac{h^{-1}\left(\frac{1}{\frac{\left(1-\frac{((2a_1-1)+1)\lambda}{1+\lambda(2a_1-1)}\right)\left(1-\frac{((2a_2-1)+1)\lambda}{1+\lambda(2a_2-1)}\right)\lambda}{\frac{((2a_1-1)+1)\lambda}{1+\lambda(2a_1-1)}\times\frac{((2a_2-1)+1)\lambda}{1+\lambda(2a_2-1)}\times(1-\lambda)}+1}\right)+1}{2} \\
&= \frac{h^{-1}\left(\frac{4a_1a_2\lambda}{1-\lambda+4a_1a_2\lambda}\right)+1}{2}
\end{aligned}$$

Notice that we have

$$\begin{aligned}
&\frac{4a_1a_2\lambda}{1-\lambda+4a_1a_2\lambda} - \lambda = \frac{\lambda(1-\lambda)(4a_1a_2-1)}{1-\lambda+4a_1a_2\lambda} \\
&\leq \frac{\lambda(1-\lambda)(4\times0.5\times0.5-1)}{1-\lambda+4a_1a_2\lambda} = 0.
\end{aligned}$$

Thus, further we have

$$\begin{aligned}
a_1 \boxplus_{CF}^{(0.5)} a_2 &= \frac{h^{-1}\left(\frac{4a_1a_2\lambda}{1-\lambda+4a_1a_2\lambda}\right)+1}{2} = \frac{\frac{\frac{4a_1a_2\lambda}{1-\lambda+4a_1a_2\lambda}-\lambda}{\lambda\left(1-\frac{4a_1a_2\lambda}{1-\lambda+4a_1a_2\lambda}\right)}+1}{2} \\
&= 2a_1a_2.
\end{aligned}$$

3) $\forall a_1 \in [0, 0.5)$, $a_2 \in [0.5, 1]$, we have

$$\begin{aligned}
&a_1 \boxplus_{CF}^{(0.5)} a_2 \\
&= \frac{h^{-1}\left(\frac{1}{\frac{(1-h(2a_1-1))(1-h(2a_2-1))\lambda}{h(2a_1-1)h(2a_2-1)(1-\lambda)}+1}\right)+1}{2} \\
&= \frac{h^{-1}\left(\frac{1}{\frac{\left(1-\frac{((2a_1-1)+1)\lambda}{1+\lambda(2a_1-1)}\right)\left(1-\frac{\lambda}{1-(1-\lambda)(2a_2-1)}\right)\lambda}{\frac{((2a_1-1)+1)\lambda}{1+\lambda(2a_1-1)}\times\frac{\lambda}{1-(1-\lambda)(2a_2-1)}\times(1-\lambda)}+1}\right)+1}{2} \\
&= \frac{h^{-1}\left(\frac{a_1\lambda}{(1-\lambda)(1-a_2)+a_1\lambda}\right)+1}{2}.
\end{aligned}$$

Let $a = \frac{a_1\lambda}{(1-\lambda)(1-a_2)+a_1\lambda}$. Then, we have

$$\begin{aligned}
&|2a_1-1| \geq |2a_2-1| \\
&\Rightarrow (2a_1-1)+(2a_2-1) \leq 0 \Rightarrow (2a_1-1)+1 \leq 1-(2a_2-1) \\
&\Rightarrow 1 \geq \frac{(2a_1-1)+1}{1-(2a_2-1)} = \frac{\frac{a_1\lambda}{(1-\lambda)(1-a_2)+a_1\lambda}}{1-\frac{a_1\lambda}{(1-\lambda)(1-a_2)+a_1\lambda}} \times \frac{1-\lambda}{\lambda} = \frac{a}{1-a} \times \frac{1-\lambda}{\lambda} \\
&\Rightarrow a \leq \lambda.
\end{aligned}$$

Similarly, $|2a_1-1| \le |2a_2-1| \Rightarrow a \ge \lambda$. Thus, further

$$a_1 \boxplus_{CF}^{(0.5)} a_2 = \frac{h^{-1}\left(\frac{a_1\lambda}{(1-\lambda)(1-a_2)+a_1\lambda}\right)+1}{2}$$

$$= \begin{cases} \frac{1}{2}\times \frac{\frac{a_1\lambda}{(1-\lambda)(1-a_2)+a_1\lambda}-\lambda}{\frac{a_1\lambda}{(1-\lambda)(1-a_2)+a_1\lambda}\times(1-\lambda)} + 0.5 \\ \quad \text{if } |2a_1-1|\le|2a_2-1| \\ \frac{1}{2}\times \frac{\frac{a_1\lambda}{(1-\lambda)(1-a_2)+a_1\lambda}-\lambda}{\lambda\times\left(1\frac{a_1\lambda}{(1-\lambda)(1-a_2)+a_1\lambda}\right)} + 0.5 \\ \quad \text{if } |2a_1-1|\ge|2a_2-1| \end{cases}$$

$$= \begin{cases} \frac{a_1+a_2-1}{1+(2a_1-1)} + 0.5 & \text{if } |2a_1-1|\le|2a_2-1| \\ \frac{a_1+a_2-1}{1-(2a_2-1)} + 0.5 & \text{if } |2a_1-1|\ge|2a_2-1| \end{cases}$$

$$= \frac{a_1+a_2-1}{1-\min\{|2a_1-1|, |2a_2-1|\}} + 0.5.$$

□

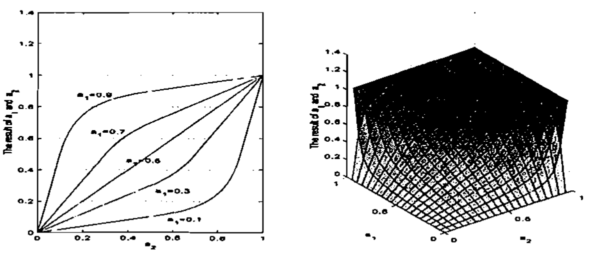


Figure 2. Contour plots of compensation operator $\boxplus_{\oplus_{CF}}$.

Corollary 3 *The following is a compensation operator:*

$$a_1 \boxplus_{\lambda}^{(0.5)} a_2 = \begin{cases} 2(a_1+a_2-a_1a_2)-1 & \textit{if } a_1>0.5,\ a_2>0.5, \\ 2a_1a_2 & \textit{if } a_1<0.5,\ a_2<0.5, \\ \frac{\sigma-\lambda}{2(1-\lambda)}+0.5 & \textit{if } \sigma\ge\lambda, \\ \frac{\sigma-\lambda}{2\lambda}+0.5 & \textit{if } \sigma<\lambda, \end{cases} \quad (12)$$

where

$$\sigma = ((2a_1'-1)(1-\lambda)+\lambda) \oplus_P (a_2'\lambda), \quad (13)$$

here $a_1', a_2' \in \{a_1, a_2\}$ *and* $a_1' \ge 0.5 \ge a_2'$.

Proof. We use Theorem 2 to prove the corollary. First, $h : [-1,1] \to [0,1]$, defined as follows, is a generator function:

$$h(x) = \begin{cases} x(1-\lambda)+\lambda & \text{if } x\ge 0, \\ (x+1)\lambda & \text{if } x<0, \end{cases} \quad (14)$$

where $\lambda \in (0,1)$. In fact, clearly $h(-1)=0$, $h(0)=\lambda$ and $h(1)=1$, and

$$x_1 \le x_2 < 0 \Rightarrow h(x_1) = (x_1+1)\lambda \le (x_2+1)\lambda = h(x_2),$$
$$x_1 \ge x_2 \ge 0 \Rightarrow h(x_1) = x_1(1-\lambda)+\lambda \ge x_2(1-\lambda)+\lambda = h(x_2),$$
$$x_1 < 0 \le x_2 \Rightarrow h(x_1) = (x_1+1)\lambda \le \lambda \le x_2(1-\lambda)+\lambda = h(x_2).$$

Therefore, according to Theorem 2, by using h and

$$h^{-1}(x) = \begin{cases} \frac{x-\lambda}{1-\lambda} & \text{if } x>\lambda, \\ \frac{x-\lambda}{\lambda} & \text{if } x\le\lambda, \end{cases}$$

we can construct the following compensation operator:

1) In the case where $a_1, a_2 \in (0.5, 1]$, we have

$$a_1 \ge 0.5 \wedge a_2 \ge 0.5 \Rightarrow 2a_1-1 \ge 0 \wedge 2a_2-1 \ge 0$$
$$\Rightarrow h(2a_1-1) \ge \lambda \wedge h(2a_2-1) \ge \lambda$$
$$\Rightarrow h(2a_1-1) \oplus_P h(2a_2-1) \ge \lambda$$
$$\Rightarrow a_1 \boxplus_{\lambda}^{(0.5)} a_2 = \frac{h^{-1}(h(2a_1-1)\oplus_P h(2a_2-1))+1}{2}$$
$$= \frac{\frac{h(2a_1-1)\oplus_P h(2a_2-1)-\lambda}{1-\lambda}+1}{2}$$
$$= \frac{\frac{\frac{1}{\frac{(1-h(2a_1-1))(1-h(2a_2-1))\lambda}{h(2a_1-1)h(2a_2-1)(1-\lambda)}+1}-\lambda}{1-\lambda}+1}{2}$$
$$= \frac{\frac{\frac{1}{\frac{(1-((2a_1-1)(1-\lambda)+\lambda))(1-((2a_2-1)(1-\lambda)+\lambda))\lambda}{((2a_1-1)(1-\lambda)+\lambda)((2a_2-1)(1-\lambda)+\lambda)(1-\lambda)}+1}-\lambda}{1-\lambda}+1}{2}$$
$$= 2(a_1+a_2-a_1a_2)-1.$$

2) In the case where $a_1, a_2 \in [0, 0.5]$, we have

$$a_1 \le 0.5 \wedge a_2 \le 0.5 \Rightarrow 2a_1-1 \le 0 \wedge 2a_2-1 \le 0$$
$$\Rightarrow h(2a_1-1) \le \lambda \wedge h(2a_2-1) \le \lambda$$
$$\Rightarrow h(2a_1-1) \oplus_P h(2a_2-1) \le \lambda$$
$$\Rightarrow a_1 \boxplus_{\lambda}^{(0.5)} a_2 = \frac{h^{-1}(h(2a_1-1)\oplus_P h(2a_2-1))+1}{2}$$
$$= \frac{\frac{h(2a_1-1)\oplus_P h(2a_2-1)-\lambda}{1-\lambda}+1}{2}$$
$$= \frac{\frac{\frac{1}{\frac{(1-h(2a_1-1))(1-h(2a_2-1))\lambda}{h(2a_1-1)h(2a_2-1)(1-\lambda)}+1}-\lambda}{\lambda}+1}{2}$$
$$= \frac{\frac{\frac{1}{\frac{(1-((2a_1-1)+1)\lambda)(1-((2a_2-1)+1)\lambda)\lambda}{((2a_1-1)+1)\lambda((2a_2-1)+1)\lambda(1-\lambda)}+1}-\lambda}{\lambda}+1}{2}$$
$$= 2a_1a_2.$$

3) In the case where $a_1', a_2' \in \{a_1, a_2\}$ and $a_1' \le 0.5 \le a_2'$,

$$a_1 \boxplus_{\lambda}^{(0.5)} a_2 = \frac{h^{-1}(h(2a_1-1)\oplus_P h(2a_2-1))+1}{2}$$
$$= \frac{h^{-1}(h(2a_1'-1)\oplus_P h(2a_2'-1))+1}{2}$$
$$= \frac{h^{-1}(((2a_1'-1)(1-\lambda)+\lambda)\oplus_P (a_2'\lambda))+1}{2}$$
$$= \frac{h^{-1}(\sigma)+1}{2} = \begin{cases} \frac{\sigma-\lambda}{2(1-\lambda)}+0.5 & \text{if } \sigma\ge\lambda, \\ \frac{\sigma-\lambda}{2\lambda}+0.5 & \text{if } \sigma<\lambda. \end{cases}$$

□

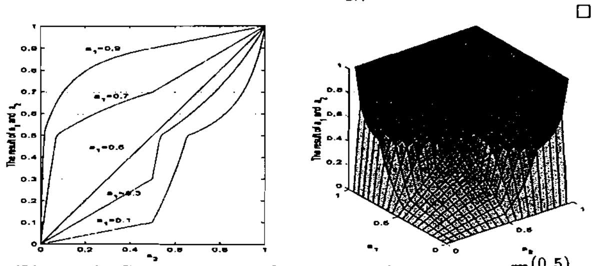


Figure 3. Contour plots of compensation operator $\boxplus_{0.1}^{(0.5)}$.

Corollary 4 *The following is a compensation operator:*

$$a_1 \overset{\sqrt{}}{\boxplus}{}_{\lambda}^{(0.5)} a_2 = \begin{cases} 0.5\left(\sqrt{\frac{\sigma_1-\lambda}{1-\lambda}}+1\right) & \text{if } a_1 > 0.5,\, a_2 > 0.5, \\ 0.5\left(1-\sqrt{\frac{\lambda-\sigma_2}{\lambda}}\right) & \text{if } a_1 < 0.5,\, a_2 < 0.5, \\ 0.5\left(\sqrt{\frac{\sigma_3-\lambda}{1-\lambda}}+1\right) & \text{if } \sigma_3 \geq \lambda, \\ 0.5\left(1-\sqrt{\frac{\lambda-\sigma_3}{\lambda}}\right) & \text{if } \sigma_3 < \lambda, \end{cases} \tag{15}$$

where

$$\sigma_1 = ((1-\lambda)(2a_1-1)^2+\lambda) \oplus_P ((1-\lambda)(2a_2-1)^2+\lambda), \tag{16}$$

$$\sigma_2 = (\lambda(1-2a_1-1)^2) \oplus_P (\lambda(1-2a_2-1)^2), \tag{17}$$

$$\sigma_3 = ((1-\lambda)(2a_1'-1)^2+\lambda) \oplus_P (\lambda(1-2a_2'-1)^2), \tag{18}$$

here $a_1', a_2' \in \{a_1, a_2\}$ *and* $a_1' \geq 0.5 \geq a_2'$.

Proof. We use Theorem 2 to prove the corollary. First, $h : [-1,1] \to [0,1]$, defined as follows, is a generator function:

$$h(x) = \begin{cases} (1-\lambda)x^2+\lambda & \text{if } x \geq 0, \\ \lambda(1-x^2) & \text{if } x < 0, \end{cases} \tag{19}$$

where $\lambda \in (0,1)$. In fact, clearly $h(-1) = 0$, $h(0) = \lambda$ and $h(1) = 1$, and

$$x_1 \leq x_2 < 0 \Rightarrow h(x_1) = -\lambda x_1^2 + \lambda \leq -\lambda x_2^2 + \lambda = h(x_2),$$

$$x_1 \geq x_2 \geq 0 \Rightarrow h(x_1) = (1-\lambda)x_1^2+\lambda \geq (1-\lambda)x_2^2+\lambda = h(x_2),$$

$$x_1 < 0 \leq x_2 \Rightarrow h(x_1) = -\lambda x_1^2+\lambda \leq \lambda \leq (1-\lambda)x_2^2+\lambda = h(x_2).$$

Therefore, according to Theorem 2, by using h and

$$h^{-1}(x) = \begin{cases} \sqrt{\frac{x-\lambda}{1-\lambda}} & \text{if } x \geq \lambda, \\ -\sqrt{\frac{\lambda-x}{\lambda}} & \text{if } x < \lambda, \end{cases}$$

we can construct the following compensation operator:

1) In the case where $a_1, a_2 \in (0.5, 1]$, we have

$$a_1 \geq 0.5 \wedge a_2 \geq 0.5 \Rightarrow 2a_1 - 1 \geq 0 \wedge 2a_2 - 1 \geq 0$$

$$\Rightarrow h(2a_1-1) \geq \lambda \wedge h(2a_2-1) \geq \lambda$$

$$\Rightarrow h(2a_1-1) \oplus_P h(2a_2-1) \geq \lambda$$

$$\Rightarrow a_1 \overset{\sqrt{}}{\boxplus}{}_{\lambda}^{(0.5)} a_2 = \frac{h^{-1}(h(2a_1-1) \oplus_P h(2a_2-1))+1}{2}$$

$$= \frac{\sqrt{\frac{h(2a_1-1)\oplus_P h(2a_2-1)-\lambda}{1-\lambda}}+1}{2}$$

$$= 0.5\left(\sqrt{\frac{\sigma_1-\lambda}{1-\lambda}}+1\right).$$

2) In the case where $a_1, a_2 \in [0, 0.5]$, we have

$$a_1 \leq 0.5 \wedge a_2 \leq 0.5 \Rightarrow 2a_1 - 1 \leq 0 \wedge 2a_2 - 1 \leq 0$$

$$\Rightarrow h(2a_1-1) \leq \lambda \wedge h(2a_2-1) \leq \lambda$$

$$\Rightarrow h(2a_1-1) \oplus_P h(2a_2-1) \leq \lambda$$

$$\Rightarrow a_1 \overset{\sqrt{}}{\boxplus}{}_{\lambda}^{(0.5)} a_2 = \frac{h^{-1}(h(2a_1-1) \oplus_P h(2a_2-1))+1}{2}$$

$$= \frac{-\sqrt{\frac{\lambda-h(2a_1-1)\oplus_P h(2a_2-1)}{\lambda}}+1}{2}$$

$$= 0.5\left(1-\sqrt{\frac{\lambda-\sigma_2}{\lambda}}\right).$$

3) In the case where $a_1', a_2' \in \{a_1, a_2\}$ and $a_1' \leq 0.5 \leq a_2'$,

$$a_1 \overset{\sqrt{}}{\boxplus}{}_{\lambda}^{(0.5)} a_2$$

$$= \frac{h^{-1}(h(2a_1-1) \oplus_P h(2a_2-1))+1}{2}$$

$$= \frac{h^{-1}(h(2a_1'-1) \oplus_P h(2a_2'-1))+1}{2}$$

$$= \frac{h^{-1}(\sigma_3)+1}{2} = \begin{cases} 0.5\left(\sqrt{\frac{\sigma_3-\lambda}{1-\lambda}}+1\right) & \text{if } \sigma \geq \lambda, \\ 0.5\left(1-\sqrt{\frac{\lambda-\sigma_3}{\lambda}}\right) & \text{if } \sigma < \lambda. \end{cases}$$

□

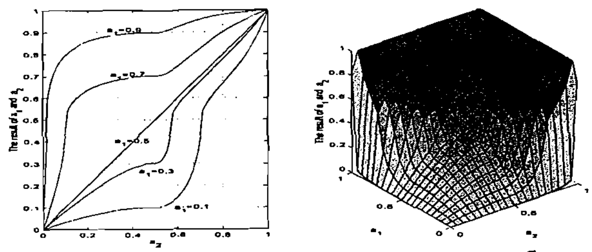


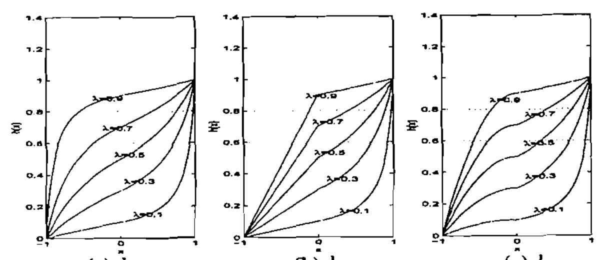
Figure 4. Contour plots of compensation operator $\overset{\sqrt{}}{\boxplus}{}_{0.1}^{(0.5)}$.

Before ending this section, let us discuss generator functions in Corollaries 4, 5, and 6. Denote generator functions (11), (14) and (19) as h_{11}, h_{22} and h_{33}, respectively. Clearly, we construct a new generator function by combining the first branch of one of these generator functions with the second branch of another among these generator functions. Thus, we obtain the following 6 new generator functions:

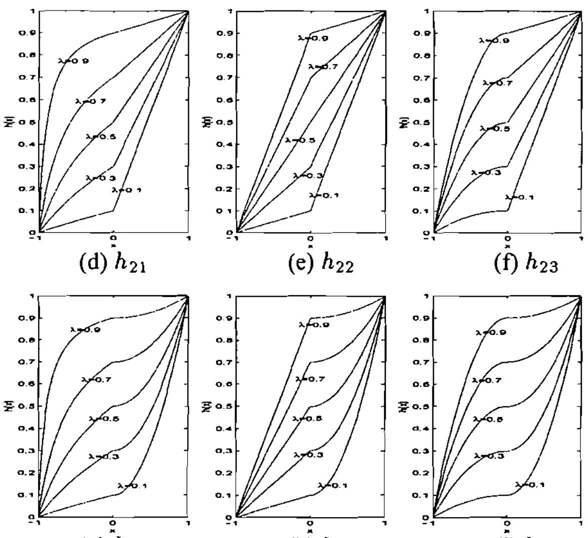
$$h_{12}(x) = \begin{cases} \frac{\lambda}{1-(1-\lambda)x} & \text{if } x \geq 0, \\ (x+1)\lambda & \text{if } x < 0; \end{cases} \quad h_{13}(x) = \begin{cases} \frac{\lambda}{1-(1-\lambda)x} & \text{if } x \geq 0, \\ \lambda(1-x^2) & \text{if } x < 0; \end{cases}$$

$$h_{21}(x) = \begin{cases} x(1-\lambda)+\lambda & \text{if } x \geq 0, \\ \frac{(x+1)\lambda}{1+\lambda x} & \text{if } x < 0; \end{cases} \quad h_{23}(x) = \begin{cases} x(1-\lambda)+\lambda & \text{if } x \geq 0, \\ \lambda(1-x^2) & \text{if } x < 0; \end{cases}$$

$$h_{31}(x) = \begin{cases} (1-\lambda)x^2+\lambda & \text{if } x \geq 0, \\ \frac{(x+1)\lambda}{1+\lambda x} & \text{if } x < 0; \end{cases} \quad h_{32}(x) = \begin{cases} (1-\lambda)x^2+\lambda & \text{if } x \geq 0, \\ \lambda(1-x^2) & \text{if } x < 0. \end{cases}$$



(a) h_{11} (b) h_{12} (c) h_{13}



(d) h_{21} (e) h_{22} (f) h_{23}

(g) h_{31} (h) h_{32} (i) h_{33}

Figure 5. Plots of some generator functions.

5 From Compensation Operators

This section uses Theorem 3 to construct compensation operators from compensation operators given in Corollaries 2-6.

Corollary 5 *For an arbitrary $\varepsilon \in (0, 1)$, the operators as follows are compensation operators:*

$$a_1 \boxplus^{(\varepsilon)}_{\boxplus_{\oplus}} a_2 = \frac{\sqrt{\beta^2 + 4\alpha\gamma_2} - \beta}{2\alpha}, \tag{20}$$

$$a_1 \boxplus^{(\varepsilon)}_{\boxplus_{CF}} a_2 = \frac{\sqrt{\beta^2 + 4\alpha\gamma_3} - \beta}{2\alpha}, \tag{21}$$

$$a_1 \boxplus^{(\varepsilon)}_{\boxplus^{(0.5)}_{\lambda}} a_2 = \frac{\sqrt{\beta^2 + 4\alpha\gamma_4} - \beta}{2\alpha}, \tag{22}$$

$$a_1 \boxplus^{(\varepsilon)}_{\overset{\surd}{\boxplus}^{(0.5)}_{\lambda}} a_2 = \frac{\sqrt{\beta^2 + 4\alpha\gamma_5} - \beta}{2\alpha}, \tag{23}$$

where

$$\alpha = \frac{\varepsilon - 0.5}{\varepsilon - \varepsilon^2}, \ \ \beta = \frac{0.5 - \varepsilon^2}{\varepsilon - \varepsilon^2},$$
$$\gamma_1 = (\alpha a_1^2 + \beta a_1) \boxplus_{\oplus} (\alpha a_2^2 + \beta a_2),$$
$$\gamma_2 = (\alpha a_1^2 + \beta a_1) \boxplus_{CF} (\alpha a_2^2 + \beta a_2),$$
$$\gamma_3 = (\alpha a_1^2 + \beta a_1) \boxplus^{(0.5)}_{\lambda} (\alpha a_2^2 + \beta a_2),$$
$$\gamma_4 = (\alpha a_1^2 + \beta a_1) \overset{\surd}{\boxplus}^{(0.5)}_{\lambda} (\alpha a_2^2 + \beta a_2).$$

Proof. We use Theorem 3 to prove the corollary. First, for an arbitrary $\varepsilon \in (0, 1)$, function $h : [0, 1] \to [0, 1]$ in Theorem 3 is defined as

$$h(x) = \alpha x^2 + \beta x, \tag{24}$$

thus,

$$h^{-1}(x) = \frac{\sqrt{\beta^2 + 4\alpha x} - \beta}{2\alpha}, \tag{25}$$

where α and β are defined as the above. Clearly, h is increasing, $h(-1) = 0$, $h(\varepsilon) = 0.5$ and $h(1) = 1$. So, for an arbitrary $\varepsilon \in (0, 1)$, by Theorem 3 we can use h and h^{-1} construct compensation operators (20), (21), (22) and (23) from compensation operators (9), (10), (12) and (15), respectively. □

6 Conclusion

The paper gives the three methods, which construct compensation operators from T-conorm-like operators, operators from the parallel combination operator, and from existing compensation operators, respectively. Moreover, a series of compensation operators are constructed by using the proposed methods. With the growing number of the operators, the study of their impact on the performance in practical application will become an important issue.

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