Risk Assessment in Natural Disasters with Fuzzy Probabilities

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Abstract

Risk assessment in regions with low earthquake activity is important for reinsurance companies and governmental building authorities. They need a complete picture of the possible risks. Even contradictive opinions have to be taken into account. Data for this kind of analysis, especially in natural disasters, is of poor quality. Standard statistical analysis is not possible. Extreme values are rare and it is therefore not possible to postulate certain distributions. We present a methodology, where we integrate expert estimates and statistical models. The result will be used for risk assessment. We demonstrate further, how to take individual and automated decisions and how to implement them efficiently.
Keywords. fuzzy probability, risk assessment, decision support, fuzzy order

1 Introduction

If we investigate natural disasters, we often realize, that the existing data is not sufficient for statistical analysis. Risk assessment on the basis of an historical earthquake catalogue for regions with low seismic activity like Germany is such an example. Instrumental measurement of earthquakes starts in China 130 BC, but it is not introduced into Europe until the late 19th century. Data before that date is a result of historians research. They used chronicles of churches and fliers, which were the standard 'news papers' in the Middle Ages

The resulting difficulties are obvious. The proper statistical model would need the characteristic data of an earthquake: the geographical position, the depth, the magnitude measured on the Richter scale, the intensity measured on the Mercalli scale. We do not have this detailed information for each event, and it has to be estimated.

The geographical position is approximated by using several sources for the same event. Magnitude and Intensity estimations are vague as we do not have any instrumental data. We might use models per region to estimate it, but we can not calculate any suitable error.

Despite the poor data quality, we have to come up with a sound result, which should serve as a basis for the calculation of premiums (reinsurance) or for a building regulations, e.g. DIN 4149 in Germany. If we would skip the none instrumental data, i.e. data before the 19th century, we would skip roughly 60% of the existing data. As a consequence we would try to estimate the risk with 300 years of experience whereas the underlying cycle is roughly 2500 years.

2 Extension of the existing Axiomatics

Modern probability theory is based on the axiomatic postulated by Kolmogorov in 1933. He postulated: **probabilistic experiment**, an experiment whose outcome depends on chance, **sample space** (Ω), the set of all possible outcomes of a random experiment, **event** (A), a subset of (Ω):

i) To every event A corresponds a non-negative number $P(A)$ with $0 \leq P(A) \leq 1$
ii) $P(\Omega)=1$
iii) Additivity property

The statistical estimates of probabilities are frequencies. In general, people omit the error calculation, who comes with the estimation. But as the outcome of the analysis depends heavily on the quality of your data, the poorer the quality the broader the confidence intervals, we can not omit the analysis for sensible analysis like risk assessment. This is especially true if we use extreme values in our analysis. We need additional information on the possible probabilities. This add on has to come from specialists and it leads to subjective probabilities.

This approach is an extension of the first axiom. We replace it with:

(1) Each event A corresponds to a function $FP(A) \in [0,1]^{[0,1]}$, $FP(A)$ is called fuzzy probability

Let us have a look on the probabilistic experiment 'throwing a dice'. The sample space is the set $\Omega = \{1, ..., 6\}$. Using a regular dice, the probability for each event is one sixth. If the sides a different by size, this is no longer valid. It is necessary to throw the dice 1000 times to get a proper estimation for the probabilities. Assuming, that time is short and you have only 30 outcomes, you have to estimate the distribution.

We assign to events A functions $FP(A)$ instead of simple numbers $P(A)$ (s. (1)). The formal assignment is shown in the paragraph 5.

Figure. 1 is an example for a simple possible assignment of fuzzy probabilities to the events like'throwing a one'.

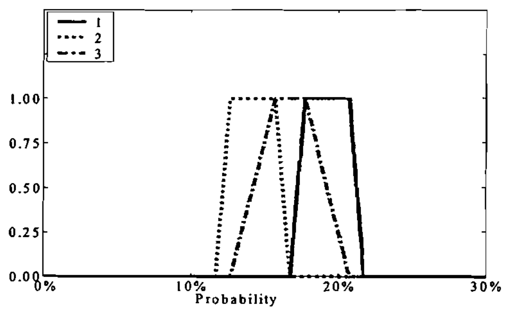

Figure. 1 Fuzzy probabilities of an irregular dice

The x-axis shows the probabilities, the y-axis the assigned possibility.

3 Embedding the existing Definition

We call probabilities according to Kolmogorov axiomatics crisp probabilities and elements of the Function space $F = [0, 1]^{[0,1]}$ fuzzy probabilities. Kolmogorov probabilities can be embedded canonical into the function space $[0, 1]^{[0,1]}$:

(2) Definition:

Suppose Ω is the sample space, F the function space $[0,1]^{[0,1]}$ and for all $A \in \Omega$ is $P(A)$ the crisp probability of the event A. We call ϕ the canonical embedding of $P(A)$ into $[0, 1]^{[0,1]}$:

$$\phi(P(A))(u) := \begin{cases} 1 & u = P(A) \\ 0 & u \neq P(A) \end{cases}$$

We use $FP(A)$ as an abbreviation of $\phi(P(A))$. The next step is to extend the operations of the sample space Ω on the function space F, by keeping the following laws:

Embedding the intersection:

(3) $A_1, A_2 \subseteq \Omega$ and $A_1 \cap A_2 = \varnothing$
$\Rightarrow P(A_1 \cap A_2) = P(A_1)P(A_2)$

Embedding the union:

(4) $A_1, A_2 \subseteq \Omega$ and $A_1 \cap A_2 = \varnothing$
$\Rightarrow P(A_1 \cup A_2) = P(A_1) + P(A_2)$

embedding the complement:

(5) $A \in \Omega \quad \Rightarrow \quad P(\neg A) = 1 - P(A)$

4 Fuzzy Probability as Fuzzy Sets

The concept of fuzzy sets was developed from Zadeh in the sixties. Fuzzy sets are generalized sub sets of an universe U. The 0-1 membership relation is replaced by a characteristic function μ. This function assigns a value of [0,1] to each element $u \in U$. The general fuzzy concepts are described in [4]. We need for our fuzzy analysis: intersection, union, complement, extension principle. Important definitions are:

(6) $\tau : [0,1]^2 \to [0,1]$ is called t - norm, if

- i) $\tau(a,1)=a$
- ii) if $b \le c$ then $\tau(a,b) \le \tau(a,c)$ monotonicity
- iii) $\tau(a,b) = \tau(b,a)$ commutativity
- iv) $\tau(a,\tau(b,c)) = \tau(\tau(a,b),c)$ associativity

(7) co –t – norm like t – Norm, but instead of i):

$\kappa(a, 0)=a$

(8) intersection:

$$\mu_{A \cap B}(u) = \tau(\mu_A(u), \mu_B(u))$$

(9) union:

$$\mu_{A \cup B}(u) = \kappa(\mu_A(u), \mu_B(u))$$

(10) $\iota:[0,1] \to [0,1]$ is called complement if

- i) $\iota(0)=1, \iota(1)=0$
- ii) (monotonicity)

We integrate fuzzy probabilities into the concept of fuzzy sets , by identifying the interval [0,1], the probabilities, as universe and FP(A) as characteristic function of the event A. The formal embedding is described in lemma (13).

We define the arithmetic for fuzzy probabilities on the basis of the theory of fuzzy set arithmetic. Intersection and union is not the appropriate method, because it is not conform with the requests (3) - (5). Let us again have a look at the example of the regular dice. Assume FP('1') is the canonic embedding of the probability of the event 'throwing a one'. The embedding must fulfill the following requirement:

$$FP('1'\cap'1')\,(u) = \begin{cases} 1 & \text{for } u = \frac{1}{16} \\ 0 & \text{otherwise} \end{cases}$$

If we calculate FP('1' $\cap$ '1')(u) with t-norm, (i.e. intersection like in (8)), we would have:

FP('1' $\cap$ '1') = FP('1') $\cap$ FP('1'), which is not the postulated result. The same is true for union and complement.

Using the extension principle (s. [4]) calculating FP($A\cap B$), FP($A\cup B$) ($A \cap B = \varnothing$) by embedding the elementary laws of probability (3) - (5):

(11) FP($A\cap B$)(x) := sup(τ(FP(A)(u),FP(B)(v)), uv=x)

(12) FP($A\cup B$)(x) := sup(τ(FP(A)(u),FP(B)(v)), u+v=x)

τ is an arbitrary t-Norm (s. (6)).

The following lemma shows the consistency of the embedding:

(13) Lemma:

for $A \cap B = \varnothing$ we have:

$\phi(P(A\cap B))$ = FP($A\cap B$) and $\phi(P(A\cup B))$ = FP($A\cup B$)

proof: τ(FP(A)(u),FP(B)(v))=1 if u=P(A) and v=P(B), 0 otherwise => lemma, same for FP($A\cup B$)

5 Estimation of Fuzzy Probabilities

The example of the dice in „Extension of the existing Axiomatics" shows the basic assignment of fuzzy probabilities for one single person. The estimation is purely subjective. It is the simplest way to analyse your data with your estimates, if the data is not rich enough for a proper statistical error calculation. The assignment shows the complete picture of the expert, it does not hide any uncertainty and we can calculate with his statement. Especially the latter two points are important.

If more than one person estimate subjective probabilities, we use the context model to aggregate their different opinions. Details of the context model are described in [1]. The principle idea is close to frequency in probability theory, the more people estimate a value, the higher the assigned possibility. We can add reliability factors by weighting the knowledge or the historical quality of past estimates. We obtain a step function as estimator. (Figure. 2)

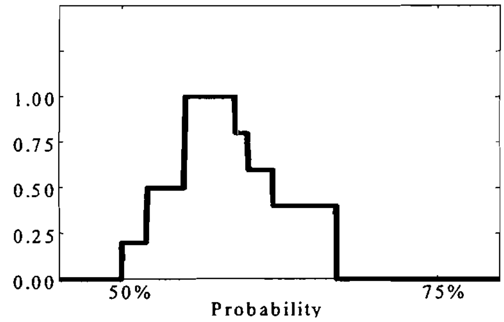

Figure. 2 Aggregated Expert Opinions

Those functions, which are typical for the of the context model, lead to an efficient implementation of fuzzy arithmetic. The standard representation of fuzzy sets is straight forward, a discreet universe and assigned possibility values. This representation is appropriate for fuzzy controllers, but it is not for models, which use fuzzy arithmetic. The problem is the calculation of the suprema and the maxima ((11), (12)),if you use max as t-Norm. We represent fuzzy sets with α - cuts and use discreet α - values:

(14) Definition:

Let A be a fuzzy set in the universe U with characteristic function μ_A. The set $A_\alpha=\{u\in U: \mu_A(u) \geq \alpha\}$ is called α - cut of fuzzy set A.

Fuzzy multiplication is faster to implement by using α - cuts due to the following lemma:

(15) Lemma

If A, B are two fuzzy sets in the universe U, μ_A, μ_B their characteristic, we have for the product:

$$(AB)_\alpha = A_\alpha\, B_\alpha$$

proof: s. [5].

The general methods for fuzzy arithmetic is found in [3]. By using lemma (15) we reduce the multiplication of fuzzy sets to interval arithmetic. α - cuts are nothing else but intervals. Even non convex fuzzy sets fit in this concept. The efficiency of the calculation depends therefore on the number of α - cuts, which is considerable lower than the number of sample points in U.

6 Semi Order on Fuzzy Probabilities

Kolmogorov-probabilities are elements of the interval [0,1]. Decision maker are able to select between different probable options, because of the total order on [0,1]. This is no longer possible if we use fuzzy probabilities. We have a natural semi order but no total order. The simplest semi order is:

(16) Definition

Let there be A, B two fuzzy probabilities, ρ a relation on $[0,1]^2$ with:

$(A,B) \in \rho \quad \Longleftrightarrow$

i) $A = B$ or

ii) $\sup\{u \mid \mu_A(u) > 0\} \leq \inf\{u \mid \mu_B(u) > 0\}$

(17) Lemma

ρ is a semi order on F. If $(A,B) \in \rho$, we write $A \leq B$.

Proof: ρ is reflexive i)

ρ is transitive: Let $(A,B) \in \rho$ and $(B,C) \in \rho$. We exclude the trivial case A, B, C are not different by pairs:

$\sup\{u \mid \mu_A(u) > 0\} \leq \inf\{u \mid \mu_B(u) > 0\}$ because $(A,B) \in \rho$

$\inf\{u \mid \mu_B(u) > 0\} \leq \sup\{u \mid \mu_B(u) > 0\}$

$\sup\{u \mid \mu_B(u) > 0\} \leq \inf\{u \mid \mu_C(u) > 0\}$ because $(B,C) \in \rho$

$\Rightarrow (A,C) \in \rho$

ρ is antisymmetric: Let $(A,B) \in \rho$ and $(B,A) \in \rho$. If $A \neq B$ we have

$\sup\{u \mid \mu_A(u) > 0\} \leq \inf\{u \mid \mu_B(u) > 0\}$ and

$\sup\{u \mid \mu_B(u) > 0\} \leq \inf\{u \mid \mu_A(u) > 0\}$

therefore $A = B$.

The implementation of this semi order is easy, because we represent fuzzy probabilities by their discreet α - cuts. Suprema and infima are simple maxima and minima of the α - cut with minimal α > 0. The disadvantage of this semi order is obvious, overlapping fuzzy probabilities are not considered. If we omit the antisymmetrie we will be able to include these cases. We use a modification of the Mabuchi index [6]. The index is calculated with the α - cuts of the underlying fuzzy sets. We extend the Mabuchi definition by a parameter ε, to focus on opinions with broad support.

(18) Definition

Let $A, B \in F$, α_i the discreet possibilities, $\varepsilon \in [0,1]$. The modified Mabuchi Index δ_ε is defined as:

$$\delta_\varepsilon: [0,1]^{[0,1]} \times [0,1]^{[0,1]} \rightarrow [-1,1]$$

$$\delta_\varepsilon = \frac{1}{a} \sum_{\alpha_i \geq \varepsilon} \alpha_i J(\alpha_i) \quad \text{with} \quad a = \sum_{\alpha_i \geq \varepsilon} \alpha_i \quad \text{and}$$

$$J(\alpha) = \frac{|\sup\{(A-B)_\alpha\}| - |\inf\{(A-B)_\alpha\}|}{|(A-B)_\alpha|}$$

The interpretation of the outcome of the index is:

$\delta_\varepsilon(A,B) = -1$ A is definitely smaller than B

$\delta_\varepsilon(A,B) = 1$ A is definitely bigger than B

By transforming δ_ε on [0,1] we interpret it as a characteristique function of the statement ‚≤', which is important for computer based decision making.

We are more interested in decision support for decision makers. Therefore we propagate the non transformed values, as they are much more intuitive as the transformed ones. It is natural to associate negative figures with '<' and positive figures with '>'. We model the style of the analysis with the parameter ε. We choose ε small for consent decisions, and ε big for broad based decisions, i.e. all possible exceptions are included.

7 Conclusion

The presented fuzzy analysis methodology enables decision makers to include softfacts like expert opinions and estimates into a statistical analysis.It is possible to include even contradictive opinions into a quantitative data analysis. This is important for risk assessment in natural disaster, because of the poor data quality.

The technical aspects like representation of fuzzy sets for fuzzy arithmetic, modified Mabuchi index allows an automated decision making and decrease the calculation costs.

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