Bootstrap Techniques: a Valuable Tool in Statistical Hypothesis Testing about the Means of Fuzzy Random Variables

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Abstract

The aim of this paper is to present in a concise and integrated way the bootstrap approach to statistical testing of hypotheses about means of fuzzy-valued random variables. We will deal with the one-, two-sample and ANOVA testing problems in a way that will allow us to see also some differences in approaching them, and also to conclude the suitability of bootstrap techniques in handling these problems.

Keywords: bootstrap, D_W^{φ}-metric, fuzzy random variables.

1 Introduction

The problem of testing hypothesis on the basis of fuzzy data has received attention in the literature (see, for instance, Casals *et al.* (3), (4), (5), Gil *et al.* (8), Grzegorzewski (11), (12), and Filzmoser and Viertl (21)). In particular, in previous recent studies (see Körner (13), Montenegro *et al.* (14), (15), (16), (18), (19), Montenegro (17)) the problem of testing statistical hypothesis on the (fuzzy) mean of a fuzzy random variable (hereafter FRV for short) has been studied. Two approaches were first considered for this aim, namely, the exact one for normal FRVs (which is a quite restrictive model in practice) and the asymptotic one. The asymptotic one-sample approach has been stated by Körner (13) who has established a limit result which is valid for general fuzzy number-valued FRVs, although this result becomes unfeasible for practical statistical purposes since it involves some unknown population parameters. On the other hand, Montenegro *et al.*'s asymptotic developments for one- and, two-sample and ANOVA's testing about means are slightly less general (in the sense that they can only be applied to fuzzy number-valued FRVs taking on a finite number of different values), but can be applied to most of real-life situations (since the assumption of a FRV taking on a finite number of different values is widely acceptable in practice).

Asymptotic tests guarantee that the larger the sample size the closer the probability of error I type to the nominal significance level is. The main inconvenience in practice is that the nominal level is only achieved for very large sample sizes. In this respect, for bootstrap procedures (although being usually based on the asymptotic ones) a faster convergence speed is expected, and hence the nominal significance level is mostly achieved for moderate and even small samples.

Based on this argument, we have recently introduced the application of bootstrap techniques in dealing with these testing problems. In fact, we have examined separately the bootstrap approximation of Montenegro *et al.*'s asymptotic results for the one-sample (see Montenegro *et al.* (15), (16), (18), Montenegro (17)), two-sample (Montenegro (17)) and ANOVA (Montenegro (17), Gil *et al.* (9)) for FRVs taking on a finite number of different values. In fact, we have indicated the immediate extension to general FRVs for the one-sample case (see (10)).

As a common conclusion, we can state that bootstrap techniques are very valuable in dealing with testing about means of FRVs since

- there are not stochastic models in the literature which can be really widely applicable to describe FRVs involved in practical situations ,
- most of the 'parameters' in general asymptotic approaches could be unknown,
- the accuracy of bootstrap approaches is greater than that of asymptotic ones for most of cases we have examined.

In this paper we summarize the main ideas in these studies so that we can note the analogies and differences in the bootstrap approaches for the three problems.

2 Preliminaries

Fuzzy random variables in Puri and Ralescu's sense (20) were introduced as an extension of random sets and to model the situations in which experiments conducted on a population of individuals assign an imprecise (fuzzy) value to each and every possible experimental outcome. Statistical methods based on Puri and Ralescu's notion in this paper will concern a fuzzy parameter: the population fuzzy-valued mean.

Let $\mathcal{F}_c(\mathbb{R})$ denote the class of normal and convex upper semicontinuous fuzzy sets of $\mathbb{R}$ with bounded closure of the support, that is, $\mathcal{F}_c(\mathbb{R}) = \{\widetilde{B} : \mathbb{R} \to [0,1] \,|\, \widetilde{B}_\alpha$ non-empty compact interval$\}$ for all $\alpha \in [0,1]\}$ where $\widetilde{B}_\alpha$ denotes the α-level set of $\widetilde{B}$ and $\widetilde{B}_0$ is the closure of the support of $\widetilde{B}$.

$\mathcal{F}_c(\mathbb{R})$ can be endowed with an inner composition law $\oplus$ extending the Minkowski addition between sets, and an external one which is the product by a scalar. These laws are compatible with the ones obtained by applying Zadeh's extension principle.

Definition 2.1 *Given the probability space $(\Omega, \mathcal{A}, P)$, a mapping $\mathcal{X} : \Omega \to \mathcal{F}_c(\mathbb{R})$ is said to be a* ***fuzzy random variable*** *associated with $(\Omega, \mathcal{A})$ if, whatever $\alpha \in [0,1]$ may be, the α-level mapping $\mathcal{X}_\alpha : \Omega \to \mathcal{B}_{\mathbb{R}}$ with*

$$\mathcal{X}_\alpha(\omega) = (\mathcal{X}(\omega))_\alpha \text{ for all } \omega \in \Omega,$$

is measurable with respect to the Borel σ-field on the class of the non-empty compact intervals associated with the Hausdorff metric on this space. We will say that FRV $\mathcal{X}$ is ***simple*** *if the cardinality of $\mathcal{X}(\Omega)$ is finite.*

Definition 2.2 *If* $\max\{|\inf \mathcal{X}_0(\cdot)|, |\sup \mathcal{X}_0(\cdot)|\} \in L^1(\Omega, \mathcal{A}, P)$*, the* ***fuzzy mean*** *of $\mathcal{X}$ is the unique fuzzy set $\widetilde{\mu} \in \mathcal{F}_c(\mathbb{R})$ such that*

$$\widetilde{\mu}_\alpha = [E(\inf \mathcal{X}_\alpha|P), E(\sup \mathcal{X}_\alpha|P)]$$

for all $\alpha \in [0,1]$.

In the class $\mathcal{F}_c(\mathbb{R})$ several metrics can be defined. For previous statistical studies concerning fuzzy random variables, a generalized metric has been shown to be especially useful.

This metric has been introduced by Bertoluzza *et al.* (2), and makes use of two weight normalized measures W and φ which can be formalized by means of two probability measures W and φ on the measurable space $([0,1], \mathcal{B}_{[0,1]})$: W is assumed to be associated with a non-degenerate distribution and φ is assumed to have a strictly increasing distribution function on $[0,1]$.

Definition 2.3 *The* (W,φ)***-distance*** *between $\widetilde{B}, \widetilde{C} \in \mathcal{F}_c(\mathbb{R})$ is defined by*

$$D_W^\varphi(\widetilde{B}, \widetilde{C}) = \sqrt{\int_{[0,1]} [d_W(\widetilde{B}_\alpha, \widetilde{C}_\alpha))]^2 d\varphi(\alpha)},$$

where d_W is defined so that

$$d_W(\widetilde{B}_\alpha, \widetilde{C}_\alpha) = \sqrt{\int_{[0,1]} \left[f_{\widetilde{B}}(\alpha,\lambda) - f_{\widetilde{C}}(\alpha,\lambda)\right]^2 \, dW(\lambda)}$$

with $f_{\widetilde{B}}(\alpha,\lambda) = \lambda \sup \widetilde{B}_\alpha + (1-\lambda) \inf \widetilde{B}_\alpha$.

3 Testing about means of FRVs: the one-sample case

Consider a probability space $(\Omega, \mathcal{A}, P)$, and let $\mathcal{X} : \Omega \to \mathcal{F}_c(\mathbb{R})$ be a FRV associated with it and taking on r different values, $\widetilde{x}_1, \ldots, \widetilde{x}_r$. Let $\widetilde{\mu}$ denote the population fuzzy mean.

For each $n \in \mathbb{N}$, consider a random sample of n independent observations from $\mathcal{X}$, and let $\overline{\mathcal{X}}$ denote the associated sample fuzzy mean and f_{ni} the random sample frequency of $\widetilde{x}_i$.

Assume that we fix a realization of the preceding random sample and we resample from it, so that a large number of samples of n independent observations from the realization is drawn, and let

$\overline{\mathcal{X}}^*$ and f^*_{ni} denote the sample fuzzy mean and the random sample frequency of $\widetilde{x}_i$ along the set of these samples, respectively. By Monte-Carlo method we can obtain the bootstrap distribution of the statistic T_n in the following result.

Method 3.1 *For a given $\widetilde{V} \in \mathcal{F}_c(\mathbb{R})$, to test at the nominal significance level $\alpha \in [0,1]$ the hypotheses $H_0 : \widetilde{\mu} = \widetilde{V}$ vs. $H_1 : \widetilde{\mu} \neq \widetilde{V}$, H_0 should be rejected whenever*

$$\frac{\left[D^{\varphi}_W(\overline{\mathcal{X}}, \widetilde{V})\right]^2}{\widehat{S}^2} > z_\alpha,$$

where z_α is the $100(1-\alpha)$ fractile of the bootstrap distribution of $T_n = \left[D^{\varphi}_W(\overline{\mathcal{X}}, \overline{\mathcal{X}}^*)\right]^2 \Big/ \widehat{S}^{*\,2}$ and with $\widehat{S}^{\,2} = \sum_{i=1}^r \left[D^{\varphi}_W(\widetilde{x}_i, \overline{\mathcal{X}})\right]^2 f_{ni}/(n-1)$, $\widehat{S}^{*\,2} = \sum_{i=1}^r \left[D^{\varphi}_W(\widetilde{x}_i, \overline{\mathcal{X}}^*)\right]^2 f^*_{ni}/(n-1)$.

Simulation studies have been developed to compare the accuracy of the bootstrap conclusions in Method 3.1 and the asymptotic ones in Montenegro *et al.* (14), (18). Simulations were inspired in our previous work (see Colubi *et al.* (6)).

We have performed simulations by choosing several values for the number of different fuzzy-values the random variables take on the population (more precisely, $r = 5$, $r = 10$ and $r = 20$; actually, we have used also $r = 50$ and $r = 100$, but the conclusions are quite similar to those assuming $r = 20$). We have also changed the distribution on these different values.

T_1 will represent the asymptotic test in Montenegro *et al.* (14), (18)). T_2 will represent the bootstrap test in Method 3.1.

For small and medium samples we have chosen different measures φ, weighting the information of the α-level sets, to calculate the distances D^{φ}_W. In this way, φ_1 will denote the uniform distribution on $[0,1]$, φ_2 will be a beta distribution $\beta(2,10)$ (i.e., the lower the level, the greater its weight) and φ_3 will be a $\beta(10,2)$ (i.e., the greater the level, the greater its weight). We have observed that, as one could expect, there is not a uniformly best choice for φ, since it depends on the shape of the fuzzy sets. For this reason, we have chosen for the large samples only φ_1.

Table 1: Simulation results for small and medium sample sizes

		T_1			T_2		
		φ_1	φ_2	φ_3	φ_1	φ_2	φ_3
$r=5$	$n=5$	16.09	16.99	23.72	2.29	2.44	2.34
	$n=10$	12	12.29	12.42	5.77	5.31	5.32
	$n=30$	6.78	6.85	7.12	4.96	4.66	4.65
	$n=100$	6.01	5.9	6.04	5.02	5.11	4.95
$r=10$	$n=5$	15.6	18.45	18.99	5.13	4.44	8.47
	$n=10$	10.42	11.34	10.79	3.85	4.73	16.21
	$n=30$	6.63	6.24	6.78	4.65	4.6	4.69
	$n=100$	5.59	5.61	5.74	4.94	4.92	4.97
$r=20$	$n=5$	23.21	20.96	20.29	6.29	5.56	6.62
	$n=10$	12.78	11.78	14.7	7.71	6.69	9.34
	$n=30$	6.59	6.61	7.7	4.61	4.6	4.94
	$n=100$	5.51	5.73	5.88	4.81	5	4.9

Table 2: Simulation results for large samples

	$r=5$		$r=10$		$r=20$	
	T_1	T_2	T_1	T_2	T_1	T_2
$n=300$	5.22	4.93	5.16	4.94	5.57	5.05
$n=500$	4.95	4.94	4.9	4.93	5.19	5.07

Table 1 summarizes the conclusions obtained, on the average, for small and medium sample sizes ($n = 5, n = 10, n = 30$, and $n = 100$), by simulating three types of fuzzy random variables, and by considering for each type two different distributions on the population values. Each simulation corresponded to 100,000 iterations and the number of bootstrap replications was 1,000. We present the percent of rejections at the significance level .05.

We have developed some simulation for large samples. The results are presented in Table 2.

We can conclude that, in general, Test T_1 works worse than T_2 for small, medium and large sample sizes.

The power functions of tests T_1 and T_2 at the significance level $\alpha = .05$ has been also discussed for a fuzzy random variable taking on 5 different values (these values being identified with S-, Z- and Π-curves, which have been obtained by a randomization process, and the 5 values having different population frequencies). 10,000 samples of size 30 have been simulated, and 1,000 bootstrap iterations have been considered. Tests T_1 and T_2 have shown a close good behavior (see Montenegro *et al.* (18) for details).

4 Testing about means of FRVs: the two-sample case

Consider two probability spaces $(\Omega_j, \mathcal{A}_j, P_j)$, $j = 1, 2$, and let $\mathcal{X}_j : \Omega_j \to \mathcal{F}_c(\mathbb{R})$ be a FRV associated with it and taking on r_j different values,

$\widetilde{x}_{j1}, \ldots, \widetilde{x}_{jr_j}$, $j = 1, 2$. Assume that $\mathcal{X}_1$ and $\mathcal{X}_2$ are independent. Let $\widetilde{\mu}_j$, $j = 1, 2$, denote the population fuzzy means. To get populations satisfying the null hypothesis to be tested, we will define the FRVs, $\mathcal{Y}_1$ and $\mathcal{Y}_2$, taking on values $\widetilde{y}_{1i_1} = \widetilde{x}_{1i_1} \oplus \overline{\mathcal{X}}_2$ and $\widetilde{y}_{2i_2} = \widetilde{x}_{2i_2} \oplus \overline{\mathcal{X}}_1$ with the corresponding respective relative frequencies.

For each $n_j \in \mathbb{N}$, consider a random sample of n_j independent observations from $\mathcal{Y}_j$, and let $\overline{\mathcal{Y}}_j$ denote the associated sample fuzzy mean and $f_{n_j i_j}$ the random sample frequency of $\widetilde{y}_{ji_j}$.

Assume that for each $j \in \{1, 2\}$ we fix a realization of the preceding random sample and we resample from it, so that a large number of samples of n_j independent observations from the realization is drawn, and let $\overline{\mathcal{Y}}_j^*$ and $f^*_{n_j i_j}$ denote the sample fuzzy mean and the random sample frequency of $\widetilde{y}_{ji_j}$ along the set of these samples, respectively.

In case the 'D_W^φ-variances' of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are assumed to coincide, then by Monte-Carlo method we can obtain the bootstrap distribution of the statistic $T^1_{n_1,n_2}$ in the following result:

Method 4.1 *To test at the nominal significance level* $\alpha \in [0,1]$ *the hypotheses* $H_0 : \widetilde{\mu}_1 = \widetilde{\mu}_2$ *vs.* $H_1 : \widetilde{\mu}_1 \neq \widetilde{\mu}_2$, H_0 *should be rejected whenever*

$$\frac{\left[D_W^\varphi(\overline{\mathcal{X}}_1, \overline{\mathcal{X}}_2)\right]^2}{\dfrac{(n_1-1)\widehat{S}_1^2}{n_1+n_2-2} + \dfrac{(n_2-1)\widehat{S}_2^2}{n_1+n_2-2}} > z_\alpha,$$

where z_α *is the* $100(1-\alpha)$ *fractile of the bootstrap distribution of*

$$T^1_{n_1,n_2} = \frac{\left[D_W^\varphi(\overline{\mathcal{Y}}_1^*, \overline{\mathcal{Y}}_2^*)\right]^2}{\dfrac{(n_1-1)\widehat{S}_1^{*\,2}}{n_1+n_2-2} + \dfrac{(n_2-1)\widehat{S}_2^{*\,2}}{n_1+n_2-2}}$$

and $\widehat{S}_j^2 = \sum_{i_j=1}^{r_j} \left[D_W^\varphi(\widetilde{x}_{ji_j}, \overline{\mathcal{X}}_j)\right]^2 f_{n_j i_j}/(n_j-1)$, $\widehat{S}_j^{*\,2} = \sum_{i_j=1}^{r_j} \left[D_W^\varphi(\widetilde{y}_{ji_j}, \overline{\mathcal{Y}}_j^*)\right]^2 f^*_{n_j i_j}/(n_j-1)$.

In case the 'D_W^φ-variances' of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ do not coincide we can follow the ideas by Babu and Singh (1), an by Monte-Carlo method we can obtain the bootstrap distribution of the statistic $T^2_{n_1,n_2}$ in the following result.

Table 3: Simulation results for the two-sample case

	$n_1=30$, $n_2=30$	$n_1=30$, $n_2=100$	$n_1=100$, $n_2=100$	$n_1=100$, $n_2=500$
M_1	6.02	7.21	5.35	5.56
M_2	5	5.85	5.04	4.97
M_3	5.02	7.24	5.11	5.13

Method 4.2 *To test at the nominal significance level* $\alpha \in [0,1]$ *the hypotheses* $H_0 : \widetilde{\mu}_1 = \widetilde{\mu}_2$ *vs.* $H_1 : \widetilde{\mu}_1 \neq \widetilde{\mu}_2$, H_0 *should be rejected whenever*

$$\frac{\left[D_W^\varphi(\overline{\mathcal{X}}_1, \overline{\mathcal{X}}_2)\right]^2}{\dfrac{\widehat{S}_1^2}{n_1} + \dfrac{\widehat{S}_2^2}{n_2}} > z_\alpha,$$

where z_α *is the* $100(1-\alpha)$ *fractile of the bootstrap distribution of*

$$T^2_{n_1,n_2} = \frac{\left[D_W^\varphi(\overline{\mathcal{Y}}_1^*, \overline{\mathcal{Y}}_2^*)\right]^2}{\dfrac{\widehat{S}_1^{*\,2}}{n_1} + \dfrac{\widehat{S}_2^{*\,2}}{n_2}}.$$

Simulation studies have been developed to compare the accuracy of the bootstrap conclusions in Method 4.2 (Test M_2), the asymptotic ones in Montenegro *et al.* (14) (Test M_1) and the real-valued one obtained by the $\lambda = .5$-average function (Test M_3). We have performed simulations from different distributions and consider 1,000 bootstrap replications, 40,000 iterations for the indicated sample sizes. The measures W and φ are the Lebesgue ones on $[0,1]$

Table 3 indicates that boostrap test M_2 from Method 4.2 improves the behavior of the asymptotic one M_1 and has a behavior similar to the one for the real-valued case M_3.

5 Testing about means of FRVs: the ANOVA test

Consider a factor which can act at J possible different levels and having fixed effects, and a response fuzzy random variable determining J independent populations. In other words, consider J probability spaces $(\Omega_j, \mathcal{A}_j, P_j)$, $j = 1, \ldots, J$, and let $\mathcal{X}_j : \Omega_j \to \mathcal{F}_c(\mathbb{R})$ be a FRV associated with it and taking on r_j different values, $\widetilde{x}_{j1}, \ldots, \widetilde{x}_{jr_j}$,

$j = 1, \ldots, J$. Assume that $\mathcal{X}_1, \ldots, \mathcal{X}_J$ are independent. Let $\widetilde{\mu}_j$, $j = 1, \ldots, J$, denote the population fuzzy means. To get bootstrap populations with a common fuzzy mean from the available sample information in this case, we can add to each sample the mean of the other samples; for this purpose, we will define the FRVs, $\mathcal{Y}_j$, taking on values

$$\widetilde{y}_{ji_j} = \widetilde{x}_{ji_j} \oplus (\overline{\mathcal{X}}_1 \oplus \ldots \oplus \overline{\mathcal{X}}_{j-1} \oplus \overline{\mathcal{X}}_{j+1} \oplus \ldots \oplus \overline{\mathcal{X}}_J)$$

with the corresponding respective relative frequencies.

For each $n_j \in \mathbb{N}$, consider a random sample of n_j independent observations from $\mathcal{Y}_j$, and let $\overline{\mathcal{Y}}_j$ denote the associated sample fuzzy mean and $f_{n_j i_j}$ the random sample frequency of $\widetilde{y}_{ji_j}$.

Assume that for each $j \in \{1, \ldots, J\}$ we fix a realization of the preceding random sample and we resample from it, so that a large number of samples of n_j independent observations from the realization is drawn, and let $\overline{\mathcal{Y}}_j^*$ and $f^*_{n_j i_j}$ denote the sample fuzzy mean and the random sample frequency of $\widetilde{y}_{ji_j}$ along the set of these samples, respectively.

By Monte-Carlo method, and based on the ideas by Babu and Singh (1) and Efron and Tibsirani (7) we can obtain the bootstrap distribution of the statistic $T_{n_1,\ldots,n_J}$ in the following result.

Method 5.1 *To test at the nominal significance level* $\alpha \in [0,1]$ *the null hypothesis* $H_0 : \widetilde{\mu}_1 = \ldots = \widetilde{\mu}_J$, H_0 *should be rejected if*

$$\frac{\displaystyle\sum_{j=1}^{J} n_j \left[D_W^{\varphi}(\overline{\mathcal{X}_j}, \overline{\mathcal{X}})\right]^2}{\displaystyle\sum_{j=1}^{J} \frac{1}{n_j} \sum_{i=1}^{n_j} \left[D_W^{\varphi}(\widetilde{x}_{ji_j}, \overline{\mathcal{X}_j})\right]^2 f_{n_j i_j}} > z_\alpha,$$

where z_α *is the* $100(1-\alpha)$ *fractile of the distribution of*

$$T_{(n_1,\ldots,n_J)} = \frac{\displaystyle\sum_{j=1}^{J} n_j \left[D_W^{\varphi}(\overline{\mathcal{Y}_j^*}, \overline{\mathcal{Y}^*})\right]^2}{\displaystyle\sum_{j=1}^{J} \frac{1}{n_j} \sum_{i=1}^{n_j} \left[D_W^{\varphi}(\widetilde{y}^*_{ji_j}, \overline{\mathcal{Y}_j^*})\right]^2 f^*_{n_j i_j}}$$

where

$$\overline{\mathcal{X}} = \frac{n_1}{n}\,\overline{\mathcal{X}_1} \oplus \ldots \oplus \frac{n_J}{n}\,\overline{\mathcal{X}_J}, \overline{\mathcal{Y}^*} = \frac{n_1}{n} \quad \overline{\mathcal{Y}_1^*} \oplus \ldots \oplus \frac{n_J}{n}\,\overline{\mathcal{Y}_J^*}.$$

In this section we have just developed some simulation studies to compare the test Method 5.1 (A_1) with the analogous (A_2) for the real-valued case when values of variables are defuzzified by means of the .5-average function.

Simulations have been reduced to the case $J = 3$, and for weighted measures W and φ coinciding with the Lebesgue measure on $[0,1]$. Each simulation corresponds to 40,000 iterations and the number of bootstrap replications was 1,000.

Table 4 assembles the obtained outputs for the percentage of rejections at the nominal significance level .05, for some different samples sizes allocations (n_j's being equal to either 30 or 100). On the basis of these simulations, we note that tests A_1 and A_2 show a quite similar behavior, and they improve for samples of the same size (as happens in the real-valued case).

Table 4: Simulations for the Bootstrap Oneway ANOVA in Real- and Fuzzy-Valued cases

n_1	30	30	30	100
n_2	30	30	100	100
n_3	30	100	100	100
A_1 (fuzzy-valued)	5.11	4.74	4.49	4.97
A_2 (real-valued)	5.09	4.69	4.48	5.01

On the other hand, Table 5 shows the power of tests A_1 and A_2 (more precisely, the percentage of rejections at the nominal significance level .05) which has been analyzed by considering the three sample sizes being equal to 30, and different '(fuzzy) deviations' from the null hypothesis. In particular,

- we first ($\widetilde{d}_0$) develop the simulation results under the null hypothesis, $\widetilde{\mu}_1 = \widetilde{\mu}_2 = \widetilde{\mu}_3$,
- we later ($\widetilde{d}_i$, $i = 1,2,3,4,5,6$) develop the simulation results under the assumption $\widetilde{\mu}_1 = \widetilde{\mu}_2$, $\widetilde{\mu}_3 = \widetilde{\mu}_1 + 0.4\, i\, \widetilde{\mu}_1$.

In Table 5 we can appreciate a clear loss of power due to the defuzzification process.

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Table 5: Simulations for the Power of the Bootstrap Oneway ANOVA

fuzzy deviation	$\widetilde{d}_0$	$\widetilde{d}_1$	$\widetilde{d}_2$	$\widetilde{d}_3$
A_1 (fuzzy-valued)	4.77	6.77	13.18	29.9
A_2 (real-valued)	4.77	5.99	8.96	15.28
fuzzy deviation	$\widetilde{d}_4$	$\widetilde{d}_5$	$\widetilde{d}_6$	
A_1 (fuzzy-valued)	57.24	88.99	99.85	
A_2 (real-valued)	24.30	37.28	50.81	

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