A note on partially ordered generalized terms

P. Eklund
M.A. Galán
Department of Computing Science
Umeå University
SE-90187 Umeå, Sweden
{peklund,magalan}@cs.umu.se

W. Gähler
Scheibenbergstr. 37
D-12685 Berlin, Germany
gaehler@rz.uni-potsdam.de

J. Medina
M. Ojeda-Aciego
A. Valverde
Dept. Matemática Aplicada
Universidad de Málaga
Málaga, Spain
{jmedina,aciego,a_valverde}@ctima.uma.es

Abstract

In this paper we study the definition of generalized powerset of terms in the context of partially ordered monads. We provide a definition of the powerset of terms over acSLAT and study some possible directions for the definition of a partially ordered term monad.

Keywords: partially ordered monad, generalised terms.

1 Introduction

Monads are (endo)functors involving a unit and a multiplication fulfilling suitable conditions. Monads are frequently considered with underlying functors over the category of unstructured sets. This is insufficient when handling objects with algebraic or topological structures. Lattices of open sets or converging filters are typical examples where generalisations towards monadic topologies ([3]) require underlying monads to be equipped with suitable lattice structures. Various lattice properties needs to be considered, in particular from the viewpoint of preserving colimits. Canonicity of order structures is seen derivable from purely functorial descriptions of inductively constructed term sets rather than from application oriented approaches to the concept of subterms.

Monads are also useful in the field of many-valued or fuzzy logic programming, for instance, extensions of the term concept towards generalised terms in form of various powersets of terms have been proposed in categorical terms; moreover, the use of distributive laws to compose well-known powerset monads over sets with the term monad, also over sets, has proven fruitful e.g. for a categorical approach to unification ([2]).

Now, the question arising is what is the behaviour of the previous setting for composing monads over sets when shifting over to composing monads over some category of structured sets, what kind of structures preserve the existing setting and which other frequently used structures will require to modify the setting. Specifically, in this paper we will be concerned with the theory of partially ordered monads and its behaviour with respect to composability of monads.

The paper is structured as follows: Section 2 introduces partially ordered monads. Then, Section 3 shows how powerset monads over sets can be extended to partially ordered monads. In Section 4, we outline problems involving underlying order structures on sets of terms. A functorial description of the conventional term monad is seen suitable for further exploration of underlying structures.

2 Partially ordered monads

In this section we consider partially ordered monads on the category acSLAT of almost complete semilattices, i.e. partially ordered sets $(X, \leq)$ such that the suprema $\sup \mathcal{M}$ of all non-empty subsets $\mathcal{M}$ of X exists. This category is general enough to include main choices of lattice structures for the purpose of many-valued logic, yet specific enough to allow for fulfillment of the requirements for partially ordered monads. Morphisms $f\colon (X, \leq) \to (Y, \leq)$ in acSLAT satisfy $f(\sup \mathcal{M}) = \sup f[\mathcal{M}]$

for non-empty subsets $\mathcal{M}$ of X.

Recall that a *monad* over a category C is a triple (φ, η, μ), where $\varphi : \mathsf{C} \to \mathsf{C}$ is a (covariant) functor, and $\eta : id \to \varphi$ and $\mu : \varphi \circ \varphi \to \varphi$ are natural transformations for which $\mu \circ \varphi\mu = \mu \circ \mu\varphi$ and $\mu \circ \varphi\eta = \mu \circ \eta\varphi = id_\varphi$ hold.

A monad (φ, η, μ) over acSLAT is said to be a *partially ordered monad* if

(i) For all mappings $f, g : Y \to \varphi X$, $f \leq g$ implies $\mu_X \circ \varphi f \leq \mu_X \circ \varphi g$, where $\leq$ is defined argumentwise with respect to the partial ordering of φX.

(ii) For each set X, $\mu_X : (\varphi\varphi X, \leq) \to (\varphi X, \leq)$ preserves non-empty suprema.

In [3], partially ordered monads over SET are introduced via *basic triples*, i.e. triples $\Phi = (\varphi, \leq, \eta)$, where $(\varphi, \leq) : \mathsf{SET} \to \mathsf{acSLAT}$, $X \mapsto (\varphi X, \leq)$ is a covariant functor, with $\varphi : \mathsf{SET} \to \mathsf{SET}$ as the underlying set functor, and $\eta : \mathrm{id} \to \varphi$ is a natural transformation. Partially ordered monads then are quadruples $\Phi = (\varphi, \leq, \eta, \mu)$ such that (φ, η, μ) is a monad satisfying conditions (i) and (ii), with φ as the underlying set functor. Moreover, partially ordered monads are used for monadic topologies, and in this context it is useful to consider underlying set functors.

3 The partially ordered powerset monad

Let L be a completely distributive lattice. We will consider the set of non-increasing mappings $A : X \to L$, i.e. mappings satisfying

$$\bigvee_{x' \geq x} A(x') = A(x) \tag{1}$$

for all $x \in X$. This set of mappings is denoted $L_\downarrow X$. The partial order on $L_\downarrow X$ is defined as $\alpha \leq \beta$, $\alpha, \beta \in L_{id} X$, meaning $\alpha(x) \leq \beta(x)$ for all $x \in X$.

For a morphism $f\colon X \to Y$ in acSLAT we define

$$L_\downarrow f(A)(y) = \bigvee_{f(x) \geq y} A(x).$$

Clearly, $L_\downarrow X$ is an object in acSLAT and $L_\downarrow f$ is a morphism in acSLAT. Further $\bigvee_{g(x) \geq y} A(x) \geq \bigvee_{g(x) \geq f(x) \geq y} A(x)$ whenever $g \geq f$, i.e. $L_\downarrow$ is increasing with respect to morphisms. Also, since f and g are morphisms in acSLAT, $L_\downarrow(f \circ g) = L_\downarrow(f) \circ L_\downarrow(g)$. Further, if $\iota_X : X \to X$ is the identity morphism, then because of (1), $L_\downarrow \iota_X = \iota_{L_\downarrow X}$. Now $L_\downarrow$ becomes a functor from acSLAT to acSLAT.

Further, define $\eta_X : X \to L_\downarrow X$ by

$$\eta_X(x)(x') = \begin{cases} 1 & \text{if } x' \leq x \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

and $\mu : L_\downarrow \circ L_\downarrow \to L_\downarrow$ by

$$\mu_X(\mathcal{M})(x) = \bigvee_{A \in L_\downarrow X} A(x) \wedge \mathcal{M}(A). \tag{3}$$

It is easy to see that η_X and μ_X are morphisms in acSLAT.

Proposition 3.1 $\mathbf{L}_\downarrow = (L_\downarrow, \eta, \mu)$ *is a partially ordered monad over* acSLAT.

Proof: In order to show that η is a natural transformation we have to prove that $Lf(\eta_X(x)) = \eta_Y(f(x))$. It is sufficient to consider the value 1. We have, on one hand, $Lf(\eta_X(x))(y) = 1$ if and only if there exists $x' \leq x$ such that $f(x') \geq y$, and on the other hand, $\eta_Y(f(x))(y) = 1$ if and only if $y \leq f(x)$. This then immediately establishes the naturality of η.

Concerning naturality of μ we will make use

$$L_\downarrow f(A)(y') = \bigvee_{f(x) \geq y'} A(x) \geq B(y') \tag{4}$$

where we choose $y = y'$ and obtain

$$\begin{aligned}
&\mu_Y(L_\downarrow L_\downarrow f(\mathcal{M}))(y) = \\
&= \bigvee_{B \in L_\downarrow Y} \bigvee_{A \in L_\downarrow X : L_\downarrow f(A) \geq B} (B(y) \wedge \mathcal{M}(A)) \\
&\overset{(4)}{\leq} \bigvee_{A \in L_\downarrow X : L_\downarrow f(A) \geq B} \bigvee_{f(x) \geq y} A(x) \wedge \mathcal{M}(A) \\
&\leq \bigvee_{A \in L_\downarrow X} \bigvee_{f(x) \geq y} (A(x) \wedge \mathcal{M}(A)) \\
&= \bigvee_{f(x) \geq y} \bigvee_{A \in L_\downarrow X} (A(x) \wedge \mathcal{M}(A)) \\
&= L_\downarrow f(\mu_X(\mathcal{M}))(y) \\
&= \bigvee_{f(x) \geq y} \mu_X(\mathcal{M})(x)
\end{aligned}$$

$$
\begin{aligned}
&= \bigvee_{A \in L_{\downarrow}X} (\mathcal{M}(A) \wedge \bigvee_{f(x) \geq y} A(x)) \\
&= \bigvee_{A \in L_{\downarrow}X} \mathcal{M}(A) \wedge L_{\downarrow}f(A)(y) \\
&\leq \bigvee_{A \in L_{\downarrow}X : L_{\downarrow}f(A) \geq B} \bigvee_{B \in L_{\downarrow}X} B(y) \wedge \mathcal{M}(A) \\
&= \bigvee_{B \in L_{\downarrow}Y} B(y) \wedge \bigvee_{A \in L_{\downarrow}X : L_{\downarrow}f(A) \geq B} \mathcal{M}(A) \\
&= \bigvee_{B \in L_{\downarrow}X} B(y) \wedge L_{\downarrow}L_{\downarrow}f(\mathcal{M})(A) \\
&= \mu_Y(L_{\downarrow}L_{\downarrow}f(\mathcal{M}))(y).
\end{aligned}
$$

Next, we need to show that $\mathbf{L}_{\downarrow}$ is a monad. Firstly, we show that $\mu \circ \eta L_{\downarrow} = \mu \circ L_{\downarrow}\eta = idL_{\downarrow}$. Indeed,

$$
\begin{aligned}
&\mu_X \circ L_{\downarrow}\eta_X(A)(x) = \\
&\quad = \bigvee_{A' \in L_{\downarrow}X} A'(x) \wedge L_{\downarrow}\eta_X(A)(A') \\
&\quad = \bigvee_{A' \in L_{\downarrow}X} A'(x) \wedge \bigvee_{x' \in X : \eta_X(x') \geq A'} A(x') \\
&\quad = \bigvee_{A' \in L_{\downarrow}X} \bigvee_{x' \in X : \eta_X(x') \geq A'} A'(x) \wedge A(x') \\
&\quad = \bigvee_{x' \in X} \eta_X(x')(x) \wedge A(x') \\
&\quad = \bigvee_{x' \geq x} A(x') \\
&\quad = A(x) \\
&\quad = id_{L_{\downarrow}X}(A)(x),
\end{aligned}
$$

and

$$
\begin{aligned}
&\mu_X \circ \eta_{L_{\downarrow}X}(A)(x) = \\
&\quad = \bigvee_{A' \in L_{\downarrow}X} A'(x) \wedge \eta_{L_{\downarrow}X}(A)(A') \\
&\quad = \bigvee_{A' \leq A} A'(x) \\
&\quad = A(x) \\
&\quad = id_{L_{\downarrow}X}(A)(x).
\end{aligned}
$$

Secondly, $\mu \circ L_{\downarrow}\mu = \mu \circ \mu L_{\downarrow}$ follows from

$$
\begin{aligned}
&\mu_X \circ L_{\downarrow}\mu_X(\mathsf{m})(x) = \\
&\quad = \bigvee_{A \in L_{\downarrow}X} A(x) \wedge \bigvee_{\mathcal{M} \in L_{\downarrow}L_{\downarrow}X : \mu_X(\mathcal{M}) \geq A} \mathsf{m}(\mathcal{M}) \\
&\quad = \bigvee_{A \in L_{\downarrow}X} \bigvee_{\mathcal{M} \in L_{\downarrow}L_{\downarrow}X : \mu_X(\mathcal{M}) \geq A} A(x) \wedge \mathsf{m}(\mathcal{M}) \\
&\quad = \bigvee_{\mathcal{M} \in L_{\downarrow}L_{\downarrow}X} \mu_X(\mathcal{M})(x) \wedge \mathsf{m}(\mathcal{M}) \\
&\quad = \bigvee_{\mathcal{M} \in L_{\downarrow}L_{\downarrow}X} (\bigvee_{A \in L_{\downarrow}X} A(x) \wedge \mathcal{M}(A)) \wedge \mathsf{m}(\mathcal{M}) \\
&\quad = \bigvee_{A \in L_{\downarrow}X} A(x) \wedge \bigvee_{\mathcal{M} \in L_{\downarrow}L_{\downarrow}X} \mathcal{M}(A) \wedge \mathsf{m}(\mathcal{M}) \\
&\quad = \bigvee_{A \in L_{\downarrow}X} A(x) \wedge \mu_{L_{\downarrow}X}(\mathsf{m})(A) \\
&\quad = \mu_X \circ \mu_{L_{\downarrow}X}(\mathsf{m})(x).
\end{aligned}
$$

Finally, we need to establish conditions (i) and (ii). Since $L_{\downarrow}$ is increasing with respect to morphisms, it is sufficient to establish condition (ii), and this condition immediately follows from (3). ■

4 A note on partially ordered terms

In the previous section we have presented an extension of the generalized powerset monad to the context of partially ordered monads over acSLAT. A similar treatment of the term monad requires a considerable level of abstraction, since the usual way of considering the set of terms cannot be adequately generalized.

Terms over SET are traditionally, in particular in computing disciplines, defined by $\Omega = \bigcup_{n=0}^{\infty} \Omega_n$ being an operator domain, where Ω_n contains n-ary operators. The term functor $T_\Omega \colon \mathsf{SET} \to \mathsf{SET}$ is then given as $T_\Omega(X) = \bigcup_{k=0}^{\infty} T_\Omega^k(X)$, where

$$
\begin{aligned}
T_\Omega^0(X) &= X, \\
T_\Omega^{k+1}(X) &= \{(n, \omega, (m_i)_{i \leq n}) \mid \omega \in \Omega_n, n \in N, \\
&\qquad m_i \in T_\Omega^k(X)\}.
\end{aligned}
$$

The term functor can be extended to a monad over SET ([4]).

The concept of subterms is frequently used for including an order structure on the set of terms. However, orderings involving subterms do not respect preservation of suprema on the underlying lattice. The provision of a useful order structure on the set of terms being a key issue, we need to describe sets of terms so as to reveal canonical order structures.

The inductive step above hides the underlying fundamental, and indeed set theoretically more strict, functorial description of the set of terms. In a functorial description, let

$$T^0_\Omega = id$$

and define

$$T^\alpha_\Omega = (\sum_{n\leq k} \Omega_n id^n) \circ \bigcup_{\beta<\alpha} T^\beta_\Omega$$

for each positive ordinal α. Finally, let

$$T_\Omega = \bigcup_{\alpha<\bar{k}} T^\alpha_\Omega$$

where $\bar{k}$ is the least cardinal greater than k and $\aleph_0$.

Such a purely functorial description of the term functor reveals a way to easily generalize the term monad provided that the categorical constructions involved are available in the underlying category. Thus, the choice of the underlying category is crucial for the problem of extending the term monad to a partially ordered monad.

As algebraic categories, such as the categories of distributive lattices or boolean algebras, are known to contain all the colimints, in particular sums, then it would seem natural to use these categories as a starting point.

So far the existence of a partially ordered extension of the term monad is an open question. It is future work to establish categorical prerequisites in order to move forward towards a partially ordered term monad $\mathbf{T}_\Omega = (T_\Omega, \eta, \mu)$ over a suitable algebraic category.

5 Future work

In [1] we showed how $\mathbf{L}_{id}$, with $L_{id}X$ consisting of all mappings $A: X \to L$, and $\mathbf{T}_\Omega$, as monads over SET, can be composed. This composed monad provides a concept of generalised terms, useful e.g. in unification theories related to many-valued logic programming [2].

For compositions of partially ordered monads, ordering relations with respect to composed functors need to be considered. Take $\Phi = (\varphi, \eta^\varphi, \mu^\varphi)$ and $\Psi = (\psi, \eta^\psi, \mu^\psi)$. Establishing the composition $\Phi \bullet \Psi$, and thus given $(\varphi X, \leq_{\varphi X})$ and $(\psi X, \leq_{\psi X})$, we need to define orderings for both $(\varphi\psi X, \leq_{\varphi\psi X})$ as well as $(\psi\varphi X, \leq_{\psi\varphi X})$. Indeed, these orderings are needed as the distributive law requires the use of a swapper $\sigma_X: \Psi\Phi X \to \Phi\Psi X$ in the defining the multiplication of the composed monad. In case of $\Phi = \mathbf{L}_\downarrow$, and in defining $\leq_{L_\downarrow \Psi X}$ and $\leq_{\Psi L_\downarrow X}$ we may expect composed ordering approaches relating to approaches found within domain theory.

The open question concerning $\mathbf{T}_\Omega$ being extendable to a partially ordered monad, and further, the composition $\mathbf{L}_\downarrow \bullet \mathbf{T}_\Omega$ also being extendable to a partially ordered monad over acSLAT is vitally important for many-valued extensions and various semantic methodologies.

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