STABILITY, INDISTINGUISHABILITY AND SMALL NUMBERS

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Abstract

We describe a model for the fuzzy sets "near zero" (small numbers) and we show how to derive some T-indistinguishabilities in the real line, which are related to the model considered in probabilistic metric spaces. We revisite the concept of stability for functional equations arising in Fuzzy Logic by using the "near zero"' model.

Keywords: small numbers, fuzzy sets, t-norm, t-conorm, strong negation, stability, strong stability, indistinguishability.

1 Introduction

This paper is motivated by the following problem: in the theory of functional equations there are several notions of stability (Moszner, 2004) but the so-called Hyers-Ulam stability has received a lot of attention. The intuitive idea of this stability is that given an equation $L(g) = R(g)$ one says that it's stable if there exists $\alpha > 0$ such that for any $\epsilon > 0$ and any function g such that

$$|L(g) - R(g)| < \epsilon$$

there is a unique solution f_0 of the equation such that

$$|g - f_0| < \alpha\epsilon,$$

where the above inequalities need to hold for all variables involved in the equation. The meaning of this is that "if a function almost satisfies the equation then there is only one solution which is close by". It seems that the theory of Fuzzy Sets is a natural framework for clarifying and/or extending this and other notions of stability. To this end we introduce fuzzy models for the description of quantities "near zero", we give some examples and characterizations and at the end we see how to clarify this motivating problem. This is also important for the consideration of functional equations involving connectives as t-norms and t-conorms as it is usually done in Fuzzy Logic. Moreover in doing this we give also some results on T-indistinguishabilities.

2 Basic definitions

Let us recall here the most basic definitions ([4], [13], [16]) and properties that we will be using in this paper.

Definition 2.1. A *t-norm* is a two-place function T from $[0,1]^2$ into $[0,1]$ such that the following conditions are satisfied for all x, x', y, y' and z in [0,1]:

(i) Associativity: $T(x, T(y, z)) = T(T(x, y), z)$;

(ii) Commutativity: $T(x, y) = T(y, x)$;

(iii) Monotonicity: $T(x, y) \leq T(x', y')$ whenever $x \leq x'$ and $y \leq y'$;

(iv) Unit element: $T(x, 1) = T(1, x) = x$;

(v) Null element: $T(x, 0) = T(0, x) = 0$.

Note that (v) follows from (iii) and (iv) and that with continuity conditions, (ii) follows from the other conditions.

The most celebrated t-norms are $\text{Min}(x, y) =$ Minimum $\{x, y\}$, $\text{Prod}(x, y) = x \cdot y$, $W(x, y) =$

Max$(x+y-1,0)$ and in the discontinuous case $Z(x,y) = 0$ if $(x,y) \in [0,1)^2$ and $Z(x,y)$ =Min(x,y) when $x=1$ or $y=1$.

Definition 2.2. A *t-conorm* is a binary operation S on [0,1] such that $S^*(x,y) = 1 - S(1-x, 1-y)$ is a t-norm.

Definition 2.3. A **strong negation** N is a continuous strictly decreasing function from [0,1] onto [0,1] such that $N(0) = 1$, $N(1) = 0$ and $N(N(x)) = N(x)$ for all x in [0,1]. Thus the classical negation is $N_0(x) = 1 - x$.

Let us quote a representation theorem (see [1]) for continuous t-norms in its latest version [4]:

Theorem 2.1. Let T be a two-place function from $[0,1]^2$ into $[0,1]$ such that:

(i) $T(x,0) = T(0,x) = 0$,

(ii) $T(1,1) = 1$,

(iii) T is associative,

(iv) T is jointly continuous.

Then T admits one of the following representations:

(a) $T(x,y) = \text{Min}(x,y)$;

(b) $T(x,y) = t^{(-1)}(t(x) + t(y))$, where t is a continuous and strictly decreasing function from $[0,1]$ into $\mathbb{R}^+$, with $t(1) = 0$ and $t^{(-1)}$ is its pseudo-inverse of t;

(c) There exists a countable collection $\{[a_n, b_n]\}$ of non-overlapping, closed, non-degenerate subintervals of $[0,1]$ and a collection of t-norms T_n each of them representable in the form (b) such that

$$T(x,y) = \begin{cases} a_n + (b_n - a_n) T_n\left(\frac{x-a_n}{b_n-a_n}, \frac{y-a_n}{b_n-a_n}\right), \\ \qquad \text{if } (x,y) \in [a_n, b_n]^2 \text{ for some } n, \\ \text{Min}(x,y), \\ \qquad \text{otherwise.} \end{cases}$$

The previous theorem yields a corresponding representation for all continuous t-conorms.

Following (Trillas, 1982) we recall the next:

Definition 2.4. Given a t-norm T an *indistinguishability operator* in a set X is a function $E : X \times X \to [0,1]$ such that for all x, y, z in X:

(1) $E(x,x) = 1$;

(2) $E(x,y) = E(y,x)$;

(3) $T(E(x,y), E(y,z)) \leq E(x,z)$.

In the case T =Prod we recover the classical definition of Menger (Menger, 1951) for an *equality relation.* When T =Min we have the *similarity relation* of Zadeh (Zadeh, 1977) and for $T = W$ we obtain Ruspini's *likeness relation* (Ruspini, 1972).

3 On small numbers

For modelling the idea of "small number"' in Fuzzy Sets Theory we state the following:

Definition 3.1. A fuzzy set $N : \mathbb{R} \to [0,1]$ will be said to model the concept of *near zero* or to be a *small fuzzy number* if it satisfies the following conditions:

(i) $N(0) = 1$;

(ii) $N(x) = N(-x)$, for all x in $\mathbb{R}$;

(iii) There exists $a > 0$ such that N is strictly decreasing in $[0,a]$ and non-increasing in $[a, \infty]$;

(iv) N is continuous.

Clearly (i) gives degree 1 to the null element 0; (ii) means that we want to have the same degrees for numbers approaching 0 either from the left or from the right; (iii) states that near zero degrees of proximity to 0 are consistent with the ordering of the real line (decreasing sequences to 0 will have increasing degrees); (iv) means that it is possible to take limits in a comfortable way.

Example 3.1. For any $k > 0$ consider

$$S_k(x) = \begin{cases} \exp(-kx), & \text{if } x \geq 0, \\ \exp(kx), & \text{if } x \leq 0. \end{cases}$$

Then all mappings S_k satisfy Definition 3.1.

Example 3.2. For any $a > 0$ consider

$$N_a(x) = \begin{cases} 0, & x \geq a, \\ 1 - x/a, & 0 \leq x \leq a, \\ 1 + x/a, & -a \leq x \leq 0, \\ 0, & x \leq a. \end{cases}$$

Then all mappings $\{N_a \mid a > 0\}$ satisfy the above Definition 3.1.

Note. We pay attention to modelling results "near zero"' because for any real z, to model the situation of getting results "near z"' one can consider $N(x \pm z)$, where N satisfies Definition 3.1, i.e., to translate the graph of $y = N(x)$ towards the point z.

4 On small numbers and indistinguishability operators

If N is a near zero fuzzy set in the above sense we can consider the fuzzy relation $E : \mathbb{R} \times \mathbb{R} \to [0, 1]$ given by

$$E_N(x, y) = N(|x - y|),$$

i.e., the degree of similarity between x and y is determined checking how their distance $|x - y|$ is close to 0. Obviously $E_N(x, x) = N(0) = 1$ and $E_N(x, y) = E_N(y, x)$, so we will study the inequality (iii) of Definition 2.4.

Theorem 4.1. Functionals E_N can't be similarities relations in the sense of Zadeh, i.e., E_N are not Min-indistinguishabilities.

Proof. If E_N would satisfy (3) with $T =$Min we would have

$$\text{Min}(N(|y - x|), N(|z - y|)) \leq N(|x - z|),$$

so if $0 < x < y < z$ and we substitute $u = y - x$ and $v = z - y$ we obtain

$$\begin{aligned} \text{Min}(N(u), N(v)) &\leq N(u + v) \leq N(\text{Max}(u, v)) \\ &= \text{Min}(N(u), N(v)), \end{aligned}$$

i.e.,

$$\text{Min}(N(u), N(v)) = N(u + v),$$

for all $u, v > 0$ so if we let $u = v$ we deduce $N(u) = N(2u)$, i.e., N would be constant, i.e., $N \equiv 1$ in contradiction with (iii) of Definition 3.1. □

Theorem 4.2. Functionals E_N are T-indistinguishabilities with T a strict t-norm generated by t if and only if there exists a subadditive function of $\mathbb{R}^+ \to \mathbb{R}^+$ such that $N = t^{-1} \circ f$.

Proof. If E_N satisfies (3) of Definition 2.4 for $T(x, y) = t^{-1}(t(x) + t(y))$ then

$$T(N(|y - x|), N(|z - y|)) \leq N(|x - z|),$$

i.e., if $0 < x < y < z$, $u = y - x$ and $v = z - y$ we will have

$$(t \circ N)(u) + (t \circ N)(v) \geq (t \circ N)(u + v),$$

so $f(z) = (t \circ N)(z)$ is a subadditive function from $\mathbb{R}^+$ into $\mathbb{R}^+$, i.e., $N = t^{-1} \circ f$ as desired. The converse is immediate. □

In particular if $T =$Prod (Menger's equality relation) then we can take $t(x) = -\ln x$ and we have $N(z) = \exp(-\alpha(x))$, which is example 3.1 when $\alpha(x) = kx$.

Theorem 4.3. Functionals E_N are T-indistinguishabilities with T a non-strict Archimedean t-norm generated by t if and only if there exists a function f such that $N(x) = (t^{-1} \circ f)(x)$, for all $x \geq 0$ and f satisfies

$$\text{Min}(f(x) + f(y), t(0)) \geq f(x + y). \qquad (1)$$

Proof. If E_N satisfies (3) of Definition 2.4 with $T(x, y) = t^{[-1]}(t(x) + t(y))$ and t bounded then (3) is equivalent to

$$(t \circ N)(u) + (t \circ N)(v) \geq (t \circ N)(u + v),$$

whenever $t(N(u)) + t(N(v)) < t(0)$. Since in all cases $(t \circ N)(z) \leq t(0)$, we have for all u, v

$$\mathrm{Min}((t \circ N)(u) + (t \circ N)(v), t(0)) \geq (t \circ N)(u+v).$$

i.e., $f(z) = t(N(z))$ satisfies (1). $\square$

In the case of Example 3.2., we have for a given $a > 0$ $E_{N_a}(x, y) = 1 - |x - y|/a$ if $|x - y| \leq a$ and $E_{N_a}(x, y) = 0$ otherwise. Therefore it is immediate to see that

$$W(E_{N_a}(x, y), E_{N_a}(y, z)) \leq E_{N_a}(x, z),$$

i.e., we have a likeness relation of Ruspini. It is easy to see that W is the strongest t-norm satisfying the previous inequality for all $x, y, z, a > 0$.

5 On fuzzy sets near zero in a metric sense

Let $N : \mathbb{R} \to [0, 1]$ be a fuzzy set "near zero"' satisfying all conditions given in Definition 3.1. We would like to consider when N satisfies the metric property:

$$N(|x - y|) + |N(x) - N(y)| = 1, \qquad (2)$$

for all x, y in an interval $[0, a]$. This equalities requires that the degree $N(|x - y|)$ of how small is the distance $|x - y|$ is complementary with the distance between the degrees of x and y.

Theorem 5.1. Under the assumptions of Definition 3.1, N satisfies $N(x) > 0$ and (2) if and only if for $x > 0$:

$$N(x) = 1 + kx$$

with $k < 0$.

Proof. If N satisfies (2) and all properties in Definition 3.1 then taking $x, y > 0$ and $z = \mathrm{Max}(x, y) - \mathrm{Min}(x, y) = |x - y|$ in $[0, a]$ we have

$$N(z) + |N(\mathrm{Min}(x, y)) - N(\mathrm{Max}(x, y)))| = 1, \quad (3)$$

then given x, y in $[0, a]$ such that $x + y \leq a$, let $u = x$ and $v = x + y$ so by (2)

$$\begin{aligned} 1 &= N(|v - u|) + |N(v) - N(u)| \\ &= N(y) + N(x) - N(x + y) \end{aligned}$$

i.e.,

$$N(x + y) - 1 = N(x) - 1 + N(y) - 1,$$

so $N(z) - 1$ is a continuous function from $\mathbb{R}^+$ into $\mathbb{R}^-$ satisfying Cauchy equation on $[0, a]$ so there exists constant k such that $N(z) - 1 = kz$, whence $N(z) = 1 + kz$ but k needs to be negative. The converse is immediate. $\square$

Note. Given a level α in (0,1) and fixed a fuzzy set "near zero"' N we will say that p is *indistinguishable from 0 modulo* α whenever

$$N(p) \geq \alpha.$$

For example, with Example 3.2, the condition $N_a(p) \geq \alpha$ gives for p in $[0, a)$, $1 - \frac{p}{a} \geq \alpha$, i.e., $p \leq a(1 - \alpha)$ and if G is a probability distribution function in Δ^+ with quasi-inverse $G^\wedge$, such that $G^\wedge(x) \leq x$ for all x in $[0, 1]$ then $pG^\wedge(x) \leq a(1 - \alpha)x = U^\wedge_{0\ a(1-a)}(x)$, whence

$$G\left(\frac{t}{p}\right) \geq U_{0,a(1-\alpha)},$$

i.e., in the probabilistic metric space associated to $\mathbb{R}$ by means of the simple structure $\mathcal{F} : \mathbb{R} \times \mathbb{R} \to \Delta^+$,

$$F_{pq}(t) = \begin{cases} G\left(\frac{t}{|p-q|}\right), & \text{if } p \neq q, \\ \epsilon_0(t), & \text{if } p = q, \end{cases}$$

we have $F_{0p} \geq U_{0,a(1-\alpha)}$, i.e., (see (Schweizer and Sklar, 1983)) p ind 0 (mod φ) where the profile function f is the uniform distribution $U_{0,a(1-\alpha)}$.

6 On the Hyers-Ulam stability for functional equations

In the theory of functional equations there are various notions of stability (Moszner, 2004). Let us consider first the following

Definition 6.1. Given a functional equation with real functions $L(f) = R(f)$ we say that it's *stable* if for any $\epsilon > 0$ there exists $\delta > 0$ such that if $|L(f) - R(f)| < \delta$ holds for all the variables involved in the equation, then there exists a solution f_0 of the equation $(L(f_0) = R(f_0))$ such that

$|f - f_0| < \epsilon$ for all variables in f and f_0. We call the equation *univocally stable* if it is stable and the solution f_0 is unique.

The Hyers-Ulam *stability* (Hyers, 1941) called *strong stability* (Moszner, 2004) for a functional equation $L(g) = R(g)$ is presented as follows:

Definition 6.2. The equation is *stable* if there exists $\alpha > 0$ such that for any $\epsilon > 0$ and for any function g such that $|L(g) - R(g)| < \epsilon$ then there is a unique solution f of the functional equation $(L(f_0) = R(f_0))$ such that $|g - f_0| < \alpha\epsilon$.

The idea is that "functions almost satisfying the equation are not far from the solution".

A natural way to face this stability notion is to locate ourselves in the framework of Fuzzy Sets Theory and use our model for fuzzy sets near zero. Let $\{N_\gamma \mid \gamma > 0\}$ be a collection of functions satisfying Definition 3.1. Then the strong stability of $L(y) = R(y)$ means that there is an $\alpha > 0$ such that for any $\epsilon > 0$ and for any function g such that

$$N_\gamma(|L(g) - R(g)|) \geq N_\gamma(\epsilon) \tag{3}$$

there is a unique solution f_0 such that

$$N_\gamma(|g - f_0|) \geq N_\gamma(\alpha\epsilon), \tag{4}$$

where inequalities (3) and (4) need to hold for all possible values of the variables involved. Nevertheless, by using appropriate functions N_γ which may vanish outside an interval, inequalities (3) and (4) may impose restricted approximations.

Example 6.1. Let T be a t-norm. We want to check the strong stability of the idempotency equation $T(x, x) = x$, for all x in [0,1] whose only solution is $T = \text{Min}$. If $\{N_\gamma \mid \gamma > 0\}$ is a collection of fuzzy sets near zero and we take any $\epsilon > 0$ then let us assume $N_\gamma(|T(x, x) - x|) \geq N_\gamma(\epsilon)$. Since N_γ is strictly decreasing for ϵ in some interval $[0, a]$ we need to have $|T(x, x) - x| \leq \epsilon$ and therefore for all x, y in [0,1]

$$\text{Min}(x, y) - T(x, y) \leq$$
$$\leq \text{Min}(x, y) - T(\text{Min}(x, y), \text{Min}(x, y)) \leq \epsilon$$

i.e., $N_\gamma(|\text{Min}(x, y) - T(x, y)|) \geq N_\gamma(\epsilon)$ and the strong stability follows.

Example 6.2. Let T be a t-norm, let S be a t-conorm, and let us consider the distributivity equation

$$S(a, T(b, c)) = T(S(a, b), S(a, c)). \tag{5}$$

The general solutions are given by $T = \text{Min}$ and S any t-conorm, so there are infinite solutions. Thus if we assume for a given $\epsilon > 0$:

$$|S(a, G(b, c)) - T(S(a, b)), S(a, c))| < \epsilon$$

substituting $b = c = 0$ we obtain $|a - T(a, a)| \leq \epsilon$ so by the previous example we deduce that necessarily $|\text{Min}(x, y) - T(x, y)| \leq \epsilon$ but we don't get any restriction on S so two very different t-conorms may be involved in the solutions, i.e., the distributivity equation is not stable in the strong sense. Nevertheless it is stable in the sense of Definition 6.2 but not universally stable: any couple (T, S) satisfying (5) is near to the solution (Min, S).

Thus in many equations with t-norms and t-conorms as logical connectives where there are infinite solutions, the solutions themselves may be faraway. Therefore the strong stability is not an appropriate concept and the weaker stability merits more attention.

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