On affine sections of 1-Lipschitz aggregation operators

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Abstract

We discuss 1-Lipschitz aggregation operators with a given affine section. The greatest 1-Lipschitz aggregation operator as well as the greatest quasi-copula with a given affine section are explicitly given. In the case of a horizontal section it is shown to be a singular copula. A similar result for the smallest (quasi-)copula with a given horizontal section is also included.

Keywords: Aggregation operator; 1-Lipschitz property; affine section

1 Introduction

Partial information about aggregation operators modelling some real situations essentially determines the operators to be used. Although in some cases this partial knowledge determines a unique aggregation operator (as in the case of strict triangular norms with known diagonal and opposite diagonal sections [4, 5]), mostly there is some degree of freedom. In the case of 1-Lipschitz aggregation operators (with no additional algebraic properties) one usually looks for the greatest and smallest solution, such as in the case of (quasi-)copulas with given diagonal section [6, 9], or if the opposite diagonal section is known [6] or if only the value in a single point is known [6, 9].

In this contribution we are concerned with 1-Lipschitz aggregation operators with a given affine section, and we derive the greatest 1-Lipschitz aggregation operator as well as the greatest quasi-copula with this property. In the case of a horizontal section, the greatest quasi-copula with this section turns out to be a singular copula. A similar result for the smallest (quasi-)copula with a given horizontal section is also given.

2 Affine sections and 1-Lipschitz aggregation operators

Consider $a, b \geq 0$ such that $a + b \leq 1$. Given an aggregation operator $A\colon [0,1]^2 \to [0,1]$, the function $\varphi_{A,a,b}\colon [0,1] \to [0,1]$ defined by

$$\varphi_{A,a,b}(x) = A(x, ax + b),$$

will be called the (a, b)-section of A.

Each (a, b)-section $\varphi_{A,a,b}$ of a 1-Lipschitz aggregation operator A is necessarily non-decreasing, $(a + 1)$-Lipschitz, and satisfies

$$\varphi_{W,a,b} \leq \varphi_{A,a,b} \leq \varphi_{W^*,a,b},$$

where the smallest and the greatest 1-Lipschitz aggregation operators are given by $W(x, y) = \max(x + y - 1, 0)$ and $W^*(x, y) = (x + y) \wedge 1$, respectively. Clearly, we also have $\varphi_{A,a,b}(0) \in [0, b]$ and $\varphi_{A,a,b}(1) \in [a + b, 1]$.

For fixed a, b as above, consider the class $\Phi_{a,b}$ of all functions $\varphi\colon [0,1] \to [0,1]$ which are non-decreasing, $(a + 1)$-Lipschitz and satisfy

$$\varphi_{W,a,b} \leq \varphi \leq \varphi_{W^*,a,b}.$$

The question arises whether for each $\varphi \in \Phi_{a,b}$ there is a 1-Lipschitz aggregation operator whose (a, b)-section coincides with φ, i.e., $\varphi_{A,a,b} = \varphi$. Note that similar problems were investigated in [6, 9] for the special case of the diagonal sections, i.e., $(1, 0)$-sections.

Theorem 2.1 *Let $\varphi \in \Phi_{a,b}$. Then the function $G^\varphi\colon [0,1]^2 \to \mathbb{R}$ defined by*

$$G^\varphi(x,y) = \begin{cases} x + \bigwedge\limits_{z\in[\max(0,\frac{y-b}{a}),x]} (\varphi(z)-z) \\ \qquad\qquad \text{if } y \le ax+b, \\ y-b+\bigwedge\limits_{z\in[x,\frac{y-b}{a}\wedge 1]} (\varphi(z)-az) \\ \qquad\qquad \text{if } y > ax+b. \end{cases} \tag{1}$$

is the greatest non-decreasing 1-Lipschitz function whose (a,b)-section coincides with φ.

Example 2.2 Put $a = 0.5$, $b = 0.3$, and let $\varphi(x) = 0.7x + 0.3$. Then $\varphi \in \Phi_{0.5,0.3}$ and G^φ is given by

$$G^\varphi(x,y) = \begin{cases} 0.7x+0.3 & \text{if } y \le 0.5x+0.3, \\ y+0.2x & \text{if } y > 0.5x+0.3. \end{cases}$$

Since $G^\varphi(0,0) = 0.3$ and $G^\varphi(1,1) = 1.2$, G^φ is not an aggregation operator.

Observe that in general we have $G^\varphi(0,0) \in [0,b]$ and $G^\varphi(1,1) \in [1, 2-a-b]$.

Theorem 2.3 *Let $\varphi \in \Phi_{a,b}$. Then the function $\overline{A}^\varphi = G^\varphi \wedge W^*$ is the greatest 1-Lipschitz aggregation operator whose (a,b)-section coincides with φ.*

Example 2.4 The greatest 1-Lipschitz aggregation operator whose $(0.5, 0.3)$-section is as in Example 2.2 is given by

$$\overline{A}^\varphi(x,y) = \begin{cases} x+y & \text{if } y \le 0.3-0.3x, \\ 0.7x+0.3 \\ \qquad \text{if } 0.3-0.3x < y \le 0.5x+0.3, \\ 0.2x+y \\ \qquad \text{if } 0.5x+0.3 < y \le 1-0.2x, \\ 1 & \text{if } y > 1-0.2x. \end{cases}$$

Corollary 2.5 *For a function $\varphi \in \Phi_{a,b}$ the function G^φ is the greatest 1-Lipschitz aggregation operator with (a,b)-section φ if and only if $\varphi(0) = 0$ and $\varphi(1) = a+b$.*

Example 2.6 Consider $a = 0$, $b \in [0,1]$ and put $\varphi(x) = W(x,b) = \max(0, x+b-1)$. Note that φ is the smallest element of $\Phi_{0,b}$, and satisfies $\varphi(0) = 0$, $\varphi(1) = b$. By Corollary 2.5, G^φ is the greatest 1-Lipschitz aggregation operator with $(0,b)$-section φ, and it is given by

$$G^\varphi(x,y) = \begin{cases} 0 \quad \text{if } (x,y) \in [0,1-b]\times[0,b], \\ x+b-1 \\ \qquad \text{if } (x,y) \in \,]1-b,1]\times[0,b], \\ y-b \\ \qquad \text{if } (x,y) \in [0,1-b]\times\,]b,1]\,, \\ x+y-1 \\ \qquad \text{if } (x,y) \in \,]1-b,1]\times\,]b,1]\,. \end{cases}$$

Note that for $b = 0$ we obtain $G^\varphi(x,y) = y$ for each $(x,y) \in [0,1]^2$, i.e., G^φ is just the projection to the second component (similarly, for $b = 1$, G^φ is the projection to the first component).

3 Consequences for quasi-copulas and copulas

Fix $a, b \ge 0$ with $a+b \le 1$. The (a,b)-section $\varphi_{Q,a,b}$ of a quasi-copula Q necessarily is non-decreasing and $(a+1)$-Lipschitz and it satisfies

$$\varphi_{W,a,b} \le \varphi_{Q,a,b} \le \varphi_{M,a,b},$$

where M is the upper Fréchet-Hoeffding bound given by $M(x,y) = x \wedge y$. Clearly, we also have $\varphi_{Q,a,b}(0) = 0$ and $\varphi_{Q,a,b}(1) = a+b$. Denote by $\Psi_{a,b}$ the class of all functions $\varphi\colon [0,1] \to [0,1]$ which are non-decreasing, $(a+1)$-Lipschitz and satisfy

$$\varphi_{W,a,b} \le \varphi \le \varphi_{M,a,b}.$$

Obviously, we have $\Psi_{a,b} \subset \Phi_{a,b}$. From Corollary 2.5 we know that, for each $\varphi \in \Psi_{a,b}$, the function G^φ defined by (1) is the greatest 1-Lipschitz aggregation operator whose (a,b)-section coincides with φ.

In this section we will look for (quasi-)copulas whose (a,b)-sections coincide with a given function $\varphi \in \Psi_{a,b}$. In general, for $\varphi \in \Psi_{a,b}$ the function G^φ need not be a quasi-copula.

Example 3.1 Put $a = 0$, $b = 0.5$ and define φ by $\varphi(x) = \frac{x}{2}$. Then $\varphi \in \Psi_{0,0.5}$ and the greatest 1-Lipschitz aggregation operator G^φ is given by

$$G^\varphi(x,y) = \begin{cases} \frac{x}{2} & \text{if } y \leq 0.5, \\ \frac{x}{2} + y - 0.5 & \text{if } y > 0.5. \end{cases}$$

Since $G^\varphi(0,1) = G^\varphi(1,0) = 0.5$, the function G^φ is not a quasi-copula.

Proposition 3.2 *Let $\varphi \in \Psi_{a,b}$. Then the function $\overline{Q}^\varphi = G^\varphi \wedge M$ is the greatest quasi-copula whose (a,b)-section coincides with φ.*

Example 3.3 The greatest quasi-copula $\overline{Q}^\varphi$ whose $(0, 0.5)$-section is φ as in Example 3.1 is given by

$$\overline{Q}^\varphi(x,y) = \begin{cases} y & \text{if } y \leq \frac{x}{2}, \\ \frac{x}{2} & \text{if } \frac{x}{2} < y \leq 0.5, \\ \frac{x}{2} + y - 0.5 & \text{if } 0.5 < y \leq \frac{x+1}{2}, \\ x & \text{if } y > \frac{x+1}{2}. \end{cases}$$

In this case, the function $\overline{Q}^\varphi$ is 2-increasing, thus it is also the greatest copula whose $(0, 0.5)$-section coincides with φ (in fact, it is a shuffle of M).

In general, the function $\overline{Q}^\varphi$ need not be a copula, compare with the case of diagonal sections [6, 9].

4 Horizontal sections

In the case of horizontal sections we have (for statistical interpretations of horizontal sections see, e.g., [8]):

Proposition 4.1 *For each $b \in [0,1]$ and $\varphi \in \Psi_{0,b}$, the function $G^\varphi \wedge M$ is the greatest copula whose $(0,b)$-section coincides with φ. Moreover, the copula $G^\varphi \wedge M$ is singular, and we have for each $(x,y) \in [0,1]^2$*

$$G^\varphi \wedge M(x,y) = (\varphi(x) \vee (\varphi(x) + y - b)) \wedge M(x,y).$$

In a similar way we can show that the smallest quasi-copula with given horizontal section is a singular copula.

Proposition 4.2 *For each $b \in [0,1]$ and each function $\varphi \in \Psi_{0,b}$, the smallest copula C_φ whose $(0,b)$-section coincides with φ is given by*

$$C_\varphi(x,y) = (\varphi(x) \wedge (\varphi(x) + y - b)) \vee W(x,y).$$

Example 4.3 Let us consider the horizontal sections of the Fréchet-Hoeffding bounds.

(i) For the function φ given by $\varphi(x) = W(x,b)$ as considered in Example 2.6, the greatest copula $G^\varphi \wedge M$ whose $(0,b)$-section coincides with φ is again a shuffle of M, and it is given by

$$(G^\varphi \wedge M)(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in [0, 1-b] \times [0,b] \cup \,]1-b, 1] \times \,]b, 1], \\ y \wedge x + b - 1 & \text{if } (x,y) \in \,]1-b, 1] \times [0,b], \\ x \wedge y - b & \text{if } (x,y) \in [0, 1-b] \times \,]b, 1]. \end{cases}$$

(ii) For the function φ given by $\varphi(x) = M(x,b)$, the smallest copula C_φ whose $(0,b)$-section coincides with φ is just the ordinal sum $(\langle 0, b, W\rangle, \langle b, 1, W\rangle)$ and, therefore, a shuffle of M.

In [8] copulas with given types of horizontal (vertical) sections are discussed: the only copula with linear horizontal sections for all $b \in [0,1]$ is the product copula Π, and copulas with quadratic horizontal sections for all $b \in [0,1]$, are only those (see [10]) which are of the form

$$C(x,y) = xy + \psi(y)x(1-x),$$

where $\psi\colon [0,1] \to [0,1]$ is a 1-Lipschitz function with $\psi(0) = \psi(1) = 0$.

Propositions 4.1 and 4.2 can be applied to determine the greatest and the smallest quasi-copula whose graphs pass through a single point $(x_0, y_0, z_0) \in [0,1]^3$ with $x_0 + y_0 - 1 \leq z_0 \leq x_0 \wedge y_0$. It turns out that these extremal quasi-copulas are singular copulas.

Lemma 4.4 *The greatest horizontal section* $\varphi^*\colon [0,1] \to [0,1]$ *and the smallest horizontal section* $\varphi_*\colon [0,1] \to [0,1]$ *of a quasi-copula* Q *satisfying* $Q(x_0, y_0) = z_0$ *with* $x_0 + y_0 - 1 \leq z_0 \leq x_0 \wedge y_0$ *are given by, respectively,*

$$\varphi^*(x) = (x \wedge z_0) \vee ((x - x_0 + z_0) \wedge y_0), \qquad (2)$$

$$\varphi_*(x) = (0 \vee (x - x_0 + z_0)) \wedge (z_0 \vee (x + y_0 - 1)). \qquad (3)$$

Remark 4.5 Let $(x_0, y_0, z_0) \in [0,1]^3$ with $x_0 + y_0 - 1 \leq z_0 \leq x_0 \wedge y_0$. The greatest quasi-copula Q^* and the smallest quasi-copula Q_* satisfying $Q^*(x_0, y_0) = Q_*(x_0, y_0) = z_0$ are singular copulas, and they are given by

$$Q^*(x,y) = (\varphi^*(x) \vee (\varphi^*(x) + y - y_0)) \wedge M(x,y),$$
$$Q_*(x,y) = (\varphi_*(x) \wedge (\varphi_*(x) + y - y_0)) \vee W(x,y),$$

where the functions φ^* and φ_* are defined by (2) and (3), respectively. Note that the formulas for Q^* and Q_* can be rewritten as follows [9]:

$$Q^*(x,y) = (z_0 + (0 \vee (x - x_0)) + (0 \vee (y - y_0))) \wedge M(x,y),$$

$$Q_*(x,y) = (z_0 + (0 \wedge (x - x_0)) + (0 \wedge (y - y_0))) \vee W(x,y).$$

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