# Basics of a formal theory of fuzzy partitions

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## Abstract

A theory of fuzzy partitions is an important part of any theory meant to provide a formal framework for fuzzy mathematics. In [3], Henkin-style higher-order fuzzy logic is introduced and proposed as a foundational theory for fuzzy mathematics. Here we investigate the properties of fuzzy partitions within its formal framework.

We follow closely the methodology of [2]. Therefore the notions introduced here are inspired by (and deduced from) the corresponding notions of classical mathematics. Sometimes they coincide with already known notions in fuzzy literature. However, we are usually more general (we work in arbitrary fuzzy logic), more expressive (we deal with the *graded* properties of fuzzy relations, as in [7]), and the proofs are more elegant (resembling the classical proofs).

**Keywords:** Fuzzy partitions, Fuzzy relations, Higher-order fuzzy logic.

## 1 Formalism

We assume that the reader is familiar with the propositional fuzzy logics with Baaz delta and their first-order variants. For convenience, we first reproduce basic definitions of higher-order fuzzy logic.

**Definition 1.1** *Let $\mathcal{F}$ be a fuzzy logic which contains Baaz $\Delta$. The Henkin-style second-order fuzzy logic over $\mathcal{F}$ is a theory over multi-sorted first-order $\mathcal{F}$ with sorts for objects (lowercase variables) and classes (uppercase variables). Both of the sorts subsume subsorts for $n$-tuples, for all $n \geq 1$. Apart from the obvious necessary function symbols and axioms for tuples (tuples equal iff their respective constituents equal), the only primitive symbol is the membership predicate $\in$ between objects and classes. The axioms for $\in$ are the following:*

1. *The comprehension axioms*
$$(\exists X)\Delta(\forall x)(x \in X \leftrightarrow \varphi),$$
*$\varphi$ not containing $X$, which enable the (eliminable) introduction of comprehension terms $\{x \mid \varphi\}$ with axioms $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$ (where $\varphi$ may be allowed to contain other comprehension terms).*

2. *The extensionality axiom*
$$(\forall x)\Delta(x \in X \leftrightarrow x \in Y) \rightarrow X = Y.$$

**Convention 1.2** *The formulae $(\forall x)(x \in X \rightarrow \varphi)$ and $(\exists x)(x \in X \mathbin{\&} \varphi)$ are abbreviated $(\forall x \in X)\varphi$ and $(\exists x \in X)\varphi$, respectively (similar notation can be used for defined binary predicates). The formulae $\varphi \mathbin{\&} \ldots \mathbin{\&} \varphi$ ($n$ times) are abbreviated $\varphi^n$. Furthermore, $x \notin X$ is shorthand for $\neg(x \in X)$, and similarly for other binary relational symbols. An alternative notation for $x \in A$ and $\langle x_1, \ldots, x_n\rangle \in R$ is simply $Ax$ and $Rx_1 \ldots x_n$, respectively.*

**Definition 1.3** *Henkin-style fuzzy logic of higher orders is obtained by repeating the previous definition on each level of the type hierarchy. Obviously, defined symbols of any type can then be shifted to all higher types as well. (Consequently, all theorems are preserved by uniform upward type-shifts.) Types may be allowed to subsume all lower types.*

*Henkin-style fuzzy logic $\mathcal{F}$ of order $n$ will be denoted by $\mathcal{F}_n$, the whole hierarchy by $\mathcal{F}_\omega$. The types of terms are either denoted by a superscripted parenthesized type (e.g., $X^{(3)}$), or understood from the context.*

It should be stressed that despite the name, Henkin-style higher-order fuzzy logics are *theories* over *first-order* fuzzy logics (see [8]). We present definitions of basic fuzzy class operations and relations.

**Definition 1.4** *In $\mathcal{F}_2$, the following elementary fuzzy set operations can be defined:*

$$\begin{aligned}
\emptyset &=_{\rm df} \{x \mid 0\}\\
\mathbf{V} &=_{\rm df} \{x \mid 1\}\\
\mathrm{Ker}(X) &=_{\rm df} \{x \mid \Delta(x \in X)\}\\
\mathrm{Supp}(X) &=_{\rm df} \{x \mid \Delta(x \notin X)\}\\
\backslash X &=_{\rm df} \{x \mid x \notin X\}\\
X \cap Y &=_{\rm df} \{x \mid x \in X \,\&\, x \in Y\}\\
X \sqcap Y &=_{\rm df} \{x \mid x \in X \wedge x \in Y\}\\
X \sqcup Y &=_{\rm df} \{x \mid x \in X \vee x \in Y\}
\end{aligned}$$

**Definition 1.5** *Further we define the following elementary relations between fuzzy sets:*

$$\begin{aligned}
\mathrm{Hgt}(X) &\equiv_{\rm df} (\exists x)(x \in X)\\
\mathrm{Norm}(X) &\equiv_{\rm df} (\exists x)\Delta(x \in X)\\
\mathrm{Crisp}(X) &\equiv_{\rm df} (\forall x)\Delta(x \in X \vee x \notin X)\\
\mathrm{Fuzzy}(X) &\equiv_{\rm df} \neg\,\mathrm{Crisp}(X)\\
\mathrm{Ext}_E(X) &\equiv_{\rm df} (\forall x, y)(Exy \,\&\, x \in X \to y \in X)\\
X \subseteq Y &\equiv_{\rm df} (\forall x)(x \in X \to x \in Y)\\
X \approx Y &\equiv_{\rm df} (\forall x)(x \in X \leftrightarrow x \in Y)\\
X \parallel Y &\equiv_{\rm df} (\exists x)(x \in X \,\&\, x \in Y)
\end{aligned}$$

We shall freely use all elementary theorems on these notions which follow from the metatheorems proved in [3], and thus can be checked by simple propositional calculations.

**Definition 1.6** *In $\mathcal{F}_2$, we define the the usual properties of relations:*

$$\begin{aligned}
\mathrm{Refl}(R) &\equiv_{\rm df} (\forall x)(Rxx)\\
\mathrm{Sym}(R) &\equiv_{\rm df} (\forall x, y)(Rxy \to Ryx)\\
\mathrm{Trans}(R) &\equiv_{\rm df} (\forall x, y, z)(Rxy \,\&\, Ryz \to Rxz)\\
\mathrm{ASym}_E(R) &\equiv_{\rm df} (\forall x, y)(Rxy \,\&\, Ryx \to Exy)
\end{aligned}$$

## 2 How to interpret the results

In the following let us assume that all fuzzy logics mentioned in the text extend Esteva and Godo's logic MTL, see [6] (the results can be extended to much weaker logic, but this is besides the scope of this short paper).

In classical fuzzy mathematics the results are usually formulated semantically. For example we know that if $X$ is a non-empty domain, $*$ a left-continuous t-norm and $R$ and $S$ reflexive binary [0,1]-fuzzy relations (i.e., $Rxx = 1$ and $Sxx = 1$ for each $x \in X$) then $R \cap S$ (where $(R \cap S)xy = Rxy * Sxy$) is reflexive relation as well.

As we already mentioned we are proving our results in formal systems of particular fuzzy logics. For example we prove the formula $\mathrm{Refl}(R) \,\&\, \mathrm{Refl}(S) \to \mathrm{Refl}(R \cap S)$ in the logic MTL. Soundness theorem gives us that for each domain $X$, each left-continuous t-norm $*$, and each binary [0,1]-fuzzy relations $R$,$S$ we have:

$$(\inf_{x\in X} Rxx) * (\inf_{x\in X} Sxx) \leq \inf_{x\in X}(Rxx * Sxx)$$

We can generalize this to arbitrary MTL-algebra (commutative bounded integral representable residuated lattice) $\mathbf{L} = (L, *, \Rightarrow, \cup, \cap, 0, 1)$ and arbitrary $\mathbf{L}$-fuzzy relations $R$,$S$.

Of course, this entails the desired known result about reflexive fuzzy relation mentioned above. This kind of semantical formulation of graded results about fuzzy relation can be found in Gottwald's book [7]. However, we need to stress that semantical formulations are just "secondary" for us. We consider [0,1] semantic (or more general algebraic one) as just "model" of fuzziness (or vagueness) captured by fuzzy logic syntactically.

## 3 Fuzzy partitions

The notion of similarity is defined as usual (in analogy with classical mathematics, it could also be called equivalence). Like all properties of fuzzy relations in our setting, it is a graded notion.

**Definition 3.1** $\mathrm{Sim}(R) \equiv_{\rm df} \mathrm{Refl}(R) \,\&\, \mathrm{Sym}(R) \,\&\, \mathrm{Trans}(R)$

Similarities are closely related with fuzzy partitions. Of course there are several ways how to define this correspondence. In this introductory paper we present definitions of two of them. The first one is more straightforward and the second one is more "fuzzy". In both cases we show the definitions and prove the basic properties only. The proper examination of these two ways of defining fuzzy partitions (and probably definition of some new ones) is to be a subject of upcoming papers.

### 3.1 The "straightforward" way:

We say that a partition corresponding to similarity $*$ is a *crisp* class of all blocks of equivalence ($[x]_\sim$), i.e., $\mathbf{V}/_\sim =_{\mathrm{df}} \{X \mid (\exists x)(X = [x]_\sim)\}$. Let $\mathcal{X}$ be class of classes resulting from some similarity in this way. By investigating properties of $\mathcal{X}$ we found four constituting properties: crispness ($\mathrm{Crisp}(\mathcal{X})$), $\Delta$-normality of its elements ($\mathrm{NormM}_\Delta(\mathcal{X})$), $\Delta$-covering ($\mathrm{Cover}_\Delta(\mathcal{X})$) and "disjointness" $\mathrm{Disj}(\mathcal{X})$. The first three properties are self-explanatory (see the formal definitions bellow), the fourth one is a straightforward (graded) generalization of the disjointness criterion that is well-known from the literature [4, 5, 10, 11].

The degree $\mathrm{Part}(\mathcal{X})$ to which a class of classes $\mathcal{X}$ is a partition is thus a straightforward (graded) generalization of the concept of a $T$-partition introduced in [4] (being put in a wider context in [5] and also appearing in [1]).

**Definition 3.2** *We define:*

- $[x]_\sim = \{y \mid y \sim x\}$
- $\mathbf{V}/_\sim =_{\mathrm{df}} \{X \mid (\exists x)(X = [x]_\sim)\}$
- $\sim_\mathcal{X} =_{\mathrm{df}} \{\langle x, y\rangle \mid (\exists X \in \mathcal{X})(x \in X \,\&\, y \in X)\}$
- $\mathrm{NormM}_\Delta(\mathcal{X}) =_{\mathrm{df}} (\forall X \in \mathcal{X})(\exists x)\Delta(x \in X)$
- $\mathrm{Cover}_\Delta(\mathcal{X}) =_{\mathrm{df}} (\forall x)(\exists X \in \mathcal{X})\Delta(x \in X)$
- $\mathrm{Disj}(\mathcal{X}) =_{\mathrm{df}} (\forall X, Y \in \mathcal{X})(X \parallel Y \to X \approx Y)$
- $\mathrm{Part}(\mathcal{X}) =_{\mathrm{df}} \mathrm{Crisp}(\mathcal{X}) \,\&\, \mathrm{NormM}_\Delta(\mathcal{X}) \,\&\, \mathrm{Cover}_\Delta(\mathcal{X}) \,\&\, \mathrm{Disj}(\mathcal{X})$

Notice the exponents in upcoming Theorems 3.3–3.5 which are caused by the non-contractivity of fuzzy logics. Semantical analogues (for t-norms only) of results of the following theorem are available in [7, Section 18.6, p. 466].

**Theorem 3.3** *It is provable in $\mathcal{F}_2$:*

1. $\mathrm{Refl}(\sim) \to (\forall x)(x \in [x]_\sim)$
2. $\mathrm{Refl}(\sim) \to (\forall x, y)([x]_\sim \approx [y]_\sim \to x \sim y)$
3. $\mathrm{Trans}(\sim) \,\&\, \mathrm{Sym}(\sim) \to (\forall x, y)(x \sim y \to [x]_\sim \approx [y]_\sim)$
4. $\mathrm{Sim}(\sim) \to (\forall x, y)([x]_\sim \approx [y]_\sim \leftrightarrow x \sim y)$

Let us recall the meaning of those graded properties. The last condition says that *the more* a relation $\sim$ is a similarity *the more* we have $(\forall x, y)([x]_\sim \approx [y]_\sim \leftrightarrow x \sim y)$. This of course entails reading usual in the above-mention papers: if relation $\sim$ is similarity, then for each $x, y$ we have: $[x]_\sim \approx [y]_\sim = x \sim y$

**Theorem 3.4** *It is provable in $\mathcal{F}_3$:*

1. $\mathrm{Crisp}(\mathbf{V}/_\sim)$
2. $\Delta\,\mathrm{Refl}(\sim) \to \mathrm{Cover}_\Delta(\mathbf{V}/_\sim)$
3. $\Delta\,\mathrm{Refl}(\sim) \to \mathrm{NormM}_\Delta(\mathbf{V}/_\sim)$
4. $\mathrm{Trans}^2(\sim) \,\&\, \mathrm{Sym}^2(\sim) \to \mathrm{Disj}(\mathbf{V}/_\sim)$
5. $\mathrm{Trans}^2(\sim) \,\&\, \mathrm{Sym}^2(\sim) \,\&\, \Delta\,\mathrm{Refl}(\sim) \to \mathrm{Part}(\mathbf{V}/_\sim)$

The last part entails known result that if relation $\sim$ is a similarity then $\mathbf{V}/_\sim$ is $T$-partition in the sense of De Baets and Mesiar.

**Theorem 3.5** *It is provable in $\mathcal{F}_3$:*

1. $\mathrm{Sym}(\sim_\mathcal{X})$
2. $\mathrm{Crisp}(\mathcal{X}) \,\&\, \mathrm{Cover}(\mathcal{X}) \to \mathrm{Refl}(\sim_\mathcal{X})$
3. $\mathrm{Disj}(\mathcal{X}) \to \mathrm{Trans}(\sim_\mathcal{X})$
4. $\mathrm{Part}(\mathcal{X}) \to \mathrm{Sim}(\sim_\mathcal{X})$

Again, the last part entails that if $\mathcal{X}$ is a $T$–partition then $\sim_{\mathcal{X}}$ is similarity. Finally, the last theorem established a correspondence between our two transformations.

**Theorem 3.6** *It is provable in $\mathcal{F}_3$:*

1. $\mathrm{Sim}(\sim) \to \sim_{\mathbf{V}/\sim} \approx \sim$
2. $\mathrm{Part}(\mathcal{X}) \to \mathbf{V}/_{\sim_{\mathcal{X}}} \approx \mathcal{X}$

### 3.2 The "fuzzy" way:

Now we try to formulate the notion of a fuzzy partition in a genuinely fuzzy way. We observe that in the definition of $\mathbf{V}/_\sim$ we mention the *cris* equality of fuzzy sets. We define a new concept of fuzzy partitions be replacing this equality by standard fuzzy equality of fuzzy sets.

Let $\mathcal{X}$ be class of classes resulting from some similarity in this way. Again, by investigating properties of $\mathcal{X}$ we found four constituting properties: extensionality w.r.t. fuzzy equality of fuzzy sets ($\mathrm{Ext}_\approx(\mathcal{X})$, normality of its elements ($\mathrm{NormM}(\mathcal{X})$), covering ($\mathrm{Cover}(\mathcal{X})$) and "disjointness" $\mathrm{Disj}(\mathcal{X})$. The last condition remains the same, however the first three changed. The resulting class of fuzzy classes is no more crisp, however it is still "behaving well" (extensionality). Notice, that in order to keep the correspondence between fuzzy partitions and similarities we need to change the definition of $\sim_{\mathcal{X}}$. However, this change is not surprising, since we are refereing twice to the same object $X$ of a *fuzzy* class $\mathcal{X}$ ($x \in X$ and $y \in X$) we should to assume its "existence" twice as well.

For the convenience and because we are not going to mix two defined notions of fuzzy partitions we will use the same denotations.

**Definition 3.7** *We define:*

- $\mathbf{V}/_\sim =_{\mathrm{df}} \{X \mid (\exists x)(X \approx [x]_\sim)\}$
- $\sim_{\mathcal{X}} =_{\mathrm{df}} \{\langle x, y\rangle \mid (\exists X)((X \in \mathcal{X})^2 \,\&\, x \in X \,\&\, y \in X)\}$
- $\mathrm{NormM}(\mathcal{X}) =_{\mathrm{df}} (\forall X \in \mathcal{X})(\exists x)(x \in X)$
- $\mathrm{Cover}(\mathcal{X}) =_{\mathrm{df}} (\forall x)(\exists X \in \mathcal{X})(x \in X)$
- $\mathrm{Disj}(\mathcal{X}) =_{\mathrm{df}} (\forall X, Y \in \mathcal{X})(X \parallel Y \to X \approx Y)$
- $\mathrm{Part}(\mathcal{X}) =_{\mathrm{df}} \mathrm{Disj}(\mathcal{X}) \,\&\, \mathrm{Cover}(\mathcal{X}) \,\&\, \mathrm{NormM}(\mathcal{X}) \,\&\, \mathrm{Ext}_\approx(\mathcal{X})$

**Theorem 3.8** *It is provable in $\mathcal{F}_3$:*

1. $\mathrm{Ext}_\approx(\mathbf{V}/_\sim)$
2. $\mathrm{Refl}(\sim) \to \mathrm{Cover}(\mathbf{V}/_\sim)$
3. $\mathrm{Refl}(\sim) \to \mathrm{NormM}(\mathbf{V}/_\sim)$
4. $\mathrm{Trans}^2(\sim) \,\&\, \mathrm{Sym}^2(\sim) \to \mathrm{Disj}(\mathbf{V}/_\sim)$
5. $\mathrm{Sim}^2(\sim) \to \mathrm{Part}(\mathbf{V}/_\sim)$

**Theorem 3.9** *It is provable in $\mathcal{F}_3$:*

1. $\mathrm{Sym}(\sim_{\mathcal{X}})$
2. $\mathrm{Cover}^2(\mathcal{X}) \,\&\, \mathrm{Ext}_\approx(\mathcal{X}) \to \mathrm{Refl}(\sim_{\mathcal{X}})$
3. $\mathrm{Disj}(\mathcal{X}) \to \mathrm{Trans}(\sim_{\mathcal{X}})$
4. $\mathrm{Part}(\mathcal{X}) \,\&\, \mathrm{Cover}(\mathcal{X}) \to \mathrm{Sim}(\sim_{\mathcal{X}})$

**Theorem 3.10** *It is provable in $\mathcal{F}_3$:*

1. $\mathrm{Sim}(\sim) \to \sim_{\mathbf{V}/\sim} \approx \sim$
2. $\mathrm{Part}(\mathcal{X}) \to \mathbf{V}/_{\sim_{\mathcal{X}}} \approx \mathcal{X}$

## Conclusion

This short paper presents (very) basic steps towards the *formal* theory of fuzzy partitions. Our motivations (in this paper) are purely formal and we are driven by theoretical analysis of the notion of partition in classical mathematics. We showed the *our* first rendering of the concept of partitions is a direct generalization of the known concept of t-partitions of B. De Baets and R. Mesiar. Then we provided more "fuzzy" version of these notions.

More thorough analysis of the defined concepts, its connection with other similar concepts defined and the literature ([9, 12, 5, 1], and finally the transition of the presented (and possible future) results to the more "applied" areas of fuzzy mathematics is a subject of a prepared joint paper with L. Běhounek and U. Bodenhofer.

**Acknowledgments**

The work was partly supported by the grant No. B100300502 of the Grant Agency of the Academy of Sciences of the Czech Republic and partly by the Institutional Research Plan AV0Z10300504.

This work is part of the research programme for formalization of fuzzy mathematics. The interested reader can check the web page of our project: www.cs.cas.cz/hp

## References

[1] R. Bělohlávek. Fuzzy Relational Systems: Foundations and Principles. New York: Kluwer Academic/Plenum Press, 2002.

[2] L. Běhounek and P. Cintula, "From fuzzy logic to fuzzy mathematics: A methodological manifesto", submitted to Fuzzy Sets Systems, 2004.

[3] L. Běhounek and P. Cintula, "Fuzzy class theory", Fuzzy Sets Systems, 154(1):34–55, 2005.

[4] B. De Baets and R. Mesiar. $T$-partitions. *Fuzzy Sets and Systems*, 97:211–223, 1998.

[5] M. Demirci. On many-valued partitions and many-valued equivalence relations. *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems*, 11(2):235–253, 2003.

[6] F. Esteva and L. Godo, Monoidal t-norm based logic: Towards a logic for left-continuous t-norms, Fuzzy Sets and Systems 123:271–288, 2001.

[7] S. Gottwald. *A Treatise on Many-Valued Logics*. Studies in Logic and Computation. Research Studies Press, Baldock, 2001.

[8] P. Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer, Dordercht, 1998.

[9] U. Höhle, Fuzzy Equalities and Indistinguishability, in Proc. 1st European Congress on Fuzzy and Intelligent Technologies 1:358-363, Aachen, 1993

[10] F. Klawonn and J. L. Castro. Similarity in fuzzy reasoning. *Mathware Soft Comput.*, 3(2):197–228, 1995.

[11] R. Kruse, J. Gebhardt, and F. Klawonn. *Foundations of Fuzzy Systems*. John Wiley & Sons, New York, 1994.

[12] H. Thiele. On Fuzzy Clusterings with Variable Boundary Conditions. *Tatra Mt. Math. Publ.*, 21:31–50, 2001.