DISTANCE AGGREGATION OPERATORS

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Summary

Based on the distance of the input set and prescribed point, the class of k–medians of Fung and Fu is recovered. A new class of generalized medians is proposed and its properties are investigated.

Keywords: aggregation operator, distance, median, minimum, maximum, k–median, (a, b)–median.

1 INTRODUCTION

Medians and their generalization play an important role in the aggregation operators theory because of their easy implementation to the ordinal case aggregation. The well–known operators min and max can be understood as medians,

$$\min(x_1,\ldots,x_n) = med(x_1,\ldots,x_n,\overbrace{0,\ldots,0}^{(n-1)-times})$$
$$\max(x_1,\ldots,x_n) = med(x_1,\ldots,x_n,\overbrace{1,\ldots,1}^{(n-1)-times}),$$

where on the right–hand sides new n-1 inputs were added. Several other types of generalized medians can be found in [8, 5].

We will discuss aggregation operators closely related to medians.

Definition 1 Aggregation operator $A : \bigcup_{n\in\mathcal{N}} [0,1]^n \to [0,1]$ is a mapping which is non–decreasing, $A(x) = x$ for each $x \in [0,1]$, and $A(0,\ldots,0) = 0$, $A(1,\ldots,1) = 1$.

It is easy to see that min is the only aggregation operator A such that for any input we have

$$A(x_1,\ldots,x_n) = d(0, A(x_1,\ldots,x_n)) \quad (1)$$
$$= d(\{0\},\{x_1,\ldots,x_n\}), \quad (2)$$

where d is Euclidean metric on the real line, and for two real subsets $E, F \subseteq \mathbb{R}$,

$$d(E,F) = \inf\{d(x,y) \mid x \in E, y \in F\}.$$

Similarly, max is the only solution of the equation

$$d(1, A(x_1,\ldots,x_n)) = d(\{1\},\{x_1,\ldots,x_n\}). \quad (3)$$

2 k–MEDIANS

We will generalize (1) and (2) for any given point $k \in [0,1]$. However, requiring (for any input)

$$d(k, A(x_1,\ldots,x_n)) = d(\{k\},\{x_1,\ldots,x_n\}), \quad (4)$$

for $0 < k < 1$ the resulting operator cannot be an aggregation operator in the sense of Definition 1 (the monotonicity is violated). Therefore we propose a modified version of (3), namely an aggregaion operator A will be called a k–*median* if it fulfills (for any input)

$$d(k, A(x_1,\ldots,x_n)) = d(\{k\},[(x_1,\ldots x_n)]), \quad (5)$$

where $[(x_1,\ldots,x_n)]$ is the smallest interval containing all input values $x_1,\ldots,x_n$.

Theorem 1 *For a given $k \in [0,1]$, there is unique k–median V_k and it is an associative idempotent operator introduced by Fung and Fu in 1975 [8].*

Proof. Recall that Fung and Fu [8], see also [6], has introduced the k–medians V_k defined by $V_k = med(x,y,k)$. Due to the associativity of V_k, for any input values $x_1,\ldots,x_n$ the resulting aggregation is uniquely determined and it can be defined by

$$V_k(x_1,\ldots,x_n) = med(x_1,\ldots,x_n,\overbrace{k,\ldots,k}^{(n-1)}), \quad (6)$$

adding new $n-1$ input values k.

It is enough to show that V_k is the unique solution of (4) in the case of two input values x, y. Then:

(i) If $x, y \in [0, k)$, $[(x, y)] = [\min(x, y), \max(x, y)]$ and hence by (4),

$$d(k, A(x,y)) = d(\{k\}, [\min(x,y), \max(x,y)] = k - \max(x,y).$$

Because of $d(k, A(k,k)) = d(\{k\}, [k,k]) = 0$, i.e., $A(k,k) = k$, and the monotonicity of A, it follows $A(x,y) = \max(x,y)$;

(ii) If $x, y \in (k, 1]$, similarly as in i), we can show that $A(x,y) = \min(x,y)$;

(iii) If $x \le k \le y$, we have $k \in [(x,y)]$ and hence by (4), $A(x,y) = k$.

Summarizing,

$$A(x,y) = \begin{cases} \max(x,y), & \text{if } x,y < k, \\ \min(x,y), & \text{if } x,y > k, \\ k, & \text{otherwise }, \end{cases}$$

i.e., $A = V_k$, see [7]. Recall that $V_0 = \min$ while $V_1 = \max$. The class $(V_k)_{k \in [0,1]}$ is just the class of all idempotent nullnorms, see [2], or equivalently, the class of all idempotent t–operations, see [10]. ∎

3 (a,b)–MEDIANS

Yager's OWA–operators [11] can be understood as a convex combination of another median–like operators, namely of $W_{n,i} : [0,1]^n \to [0,1]$, $n \in \mathcal{N}, i \in \{1, \ldots, n\}$, with

$$W_{n,i}(x_1, \ldots, x_n) = med(x_1, \ldots, x_n, 0, \ldots, 0, 1, \ldots, 1), \tag{7}$$

where $n - i$ new zero inputs and $i - 1$ new one inputs are added. It is easy to see that

$$W_{n,1} = \min \text{ and } W_{n,n} = \max.$$

The next proposed operators generalize both (5) and (6).

Definition 2 For given $a, b \in [0,1], a \le b, n \in \mathcal{N}, i \in \{1, \ldots, n\}$, the operator $V_{a,b,n,i} : [0,1]^n \to [0,1]$ defined by

$$V_{a,b,n,i}(x_1, \ldots, x_n) = med(x_1, \ldots, x_n, a, \ldots, a, b, \ldots, b), \tag{8}$$

where $n - i$ new a inputs and $i - 1$ new b inputs are added, will be called an (a,b,n,i)–median.

By means of (a,b,n,i)–medians we can define a new class of OWA–like operators, so called MEOWA-operators,

$$MW_{a,b}(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_{n,i} \cdot V_{a,b,n,i}(x_1, \ldots, x_n). \tag{9}$$

Note the $MW_{0,1}$ gives the standard OWA–operators, while $MW_{k,k} = V_k$. More details will be given in a forthcoming paper [4].

Now, we give some properties of (a,b,n,i)–medians which are idempotent, symmetric continuous n–ary aggregation operators:

1. $V_{a,b,n,i} = V_{b,a,n,n-i+1}$,
2. $V_{0,1,n,i} = W_{n,i}$,
3. $V_{k,k,n,i} = V_k / [0,1]^n$,
4. $V_{a,b,n,1} = V_a / [0,1]^n$,
5. $V_{a,b,n,n} = V_b / [0,1]^n$,
6. $d([(a,b)], V_{a,b,n,i}(x_1, \ldots, x_n)) = d([(a,b)], [(x_1, \ldots, x_n)]$.

4 EXAMPLE

Take as an input bag the triple $(0.8, 0.4, 0.6)$. Because of the idempotency of proposed operators, we have always $V_{a,b,3,i} \in [0.4, 0.8]$. The minimal output 0.4 is obtained whenever $a, b \le 0.4$, or $a \le 0.4, b \ge 0.4, i = 1$, or $a \ge 0.4, b \le 0.4, i = 3$. Similarly, the maximal output 0.8 is obtained whenever $a, b \ge 0.8$, or $a \ge 0.8, b \le 0.8, i = 1$, or $a \le 0.8, b \ge 0.8, i = 3$. The medain output is obtained whenever $a = b = 0.6$ or $a \le 0.6, b \ge 0.6, i = 2$. However, any value $c \in [0.4, 0.8]$ can be obtained by means of $V_{a,b,3,i}(0.8, 0.4, 0.6)$, e. g., $V_{c,c,3,i}(0.8, 0.4, 0.6) = V_c(0.8, 0.4, 0.6) = c$, for any $i \in \{1, 2, 3\}$.

5 CONCLUSIONS

We have characterizated by means of distance the class of k–medians. More, inspired by several median–like operators, we have introduced a new class of (a,b,n,i)–medians generalizing both k–medians and basic OWA–operators $W_{n,i}$. Proposed operators can be used for defining a new type of OWA–like operators. More, their median background seems to be promising in the new field of ordinal aggregation operators.

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