On the Equivalence between Distances and T-Indistinguishabilities *

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Summary

This paper deals with the relationship between indistinguishability operators with respect to semigroups $\mathcal{T} = ([0,1], T, \leq; 1)$, where T is a t-norm, and generalized distances with respect to the semigroup $(\mathbb{R}^+, +, \leq; 0)$, that is, ordinary distances.

The paper is organized as follows. Section 2 establishes results for ordinary distances when considered as a particular case of generalized distances. Section 3 continues the work presented in [6] but in a more general framework that allows to obtain distances not only from Prod-indistinguishabilities but also from any T_φ-indistinguishability, being T a t-norm and $T_\varphi \in \mathcal{F}(T)$. This framework also allows to build T_φ-indistinguishabilities from distances. Finally, section 4 shows that, when T is an archimedean t-norm, the proposed framework provides a characterization of T_φ-indistinguishabilities based on the t-norm additive generator. When $\varphi = Id$, this result generalizes the Menger's Prod-indistinguishabilities characterization ([2]) and recovers the results found in [1].

Keywords: distances, indistinguishabilities, t-norms, archimedean t-norms.

1 Introduction

1.1. In 1967, E. Trillas introduced (see [3] and [5]) the concept of generalized distance as a function that assigns to each pair of elements from a given set a value in an ordered and commutative semigroup. More precisely, if $\mathcal{S} = (S, *, \leq; e)$ is such a semigroup with neutral element e, X is a non-empty set and $d : X \times X \to S$ is a function that verifies:

M1) $d(x,x) = e$
M2) $d(x,y) = d(y,x)$
M3) $d(x,z) \leq d(x,y) * d(y,z)$

for any $x, y, z \in X$, then d is called a **$\mathcal{S}$-generalized distance on** X (see [5] and [3]). Both the pseudo-distances and the distances defined on $\mathbb{R}^+$ by means of the operation $+$, the linear order $\leq$ and $e = 0$ are examples of generalized distances.

On the other hand, if $\mathcal{S} = (S, *, \leq; s)$ is a commutative ordered semigroup with a distinguished element s, a function $I : X \times X \to S$ is called an $\mathcal{S}$-**indistinguishability operator** of level s for the set X if, for any $x, y, z \in X$, it verifies (see [4] and [7]):

I1) $I(x,x) \geq s$
I2) $I(x,y) = I(y,x)$
I3) $I(x,y) * I(y,z) \leq I(x,z)$

If $S = [0,1]$, T is a t-norm and T^* its dual t-conorm, that is $T^*(x,y) = 1 - T(1-x, 1-y)$, then both $\mathcal{T} = ([0,1], T, \leq; 1)$ and $\mathcal{T}^* = ([0,1], T^*, \leq; 0)$ are commutative ordered semigroups. When I is a 1-level indistinguishability operator with respect to $\mathcal{T}$, the first and third axioms become: I1) $I(x,x) = 1$; I3) $T(I(x,y), I(y,z)) \leq I(x,z)$; and I is simply said to be a T-indistinguishability. When d is a generalized distance with respect to $\mathcal{T}^*$, the first and third distances axioms become: M1) $d(x,x) = 0$ and M3) $d(x,z) \leq T^*(d(x,y), d(y,z))$.

[6] was devoted to investigate the relations between generalized distances with respect to the semigroup $(\mathbb{R}^+, +, \leq; 0)$ (which will be from now on called ordinary distances) and Prod-indistinguishabilities. The aforementioned paper makes use of the two following families of functions: the set M_1, made of all the continuous and strictly decreasing functions $t : [0,1] \to \mathbb{R}^+ = [0, +\infty]$ such that $t(1) = 0$ and $t(x \cdot y) \leq t(x) + t(y)$; and the set M_2 of all the continuous and strictly decreasing functions $s : \mathbb{R}^+ \to [0,1]$ that verify $s(0) = 1$ and $s(x) \cdot s(y) \leq s(x+y)$. These functions allow to obtain, respectively, distances from Prod-indistinguishabilities and vice versa, and recover Menger's Prod-indistinguishabilities characterization theorem (see [2]).

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1.2. Some basic results regarding t-norms should also be reviewed. As it is well-known, any t-norm T verifies $T \leq \text{Min}$ and, as a consequence, $T(x,x) \leq x$ for any $x \in [0,1]$; the boundary conditions $T(1,1) = 1$ and $T(0,0) = 0$ also hold. If T is a continuous function, then T belongs to one and only one of the following classes:

(1) $T(x,x) = x$ for any $x \in [0,1]$, and T is said to be *idempotent*.
(2) $T(x,x) < x$ for any $x \in (0,1)$, and T is said to be an *archimedean t-norm*.
(3) There exist x, y in $(0,1)$ such that $T(x,x) = x$, $T(y,y) < y$, and T is said to be a *non-idempotent and non-archimedean t-norm*.

Class (1) reduces to $T = \text{Min}$. Class (2) can be divided into two different subclasses:

(2.1) T has zero divisors, that is, there exist $x, y \in (0,1]$ such that $T(x,y) = 0$. In this case T is said to be a *non-strict archimedean t-norm* or a *nilpotent t-norm*;
(2.2) T has not zero divisors, that is, for any $x, y \in (0,1]$ it is $T(x,y) > 0$, and T is said to be a *strict archimedean t-norm*.

In general, if T is a t-norm, the family of T, $\mathcal{F}(T)$, is the set of all t-norms T_φ such that $T_\varphi = \varphi^{-1} \circ T \circ (\varphi \times \varphi)$, where $\varphi : [0,1] \to [0,1]$ is any strictly increasing bijection such that $\varphi(0) = 0$ and $\varphi(1) = 1$, that is, any order automorphism of the unit interval. $\mathcal{F}(T)$ is known as the T-family of t-norms.

Subclass (2.1) is made of all continuous t-norms that may be written as $T = \varphi^{-1} \circ W \circ (\varphi \times \varphi)$, where $W(x,y) = \text{Max}(0, x+y-1)$ is the Łukasiewicz t-norm. This subclass is the *Łukasiewicz family*, $\mathcal{F}(W)$.

Subclass (2.2) contains all continuous t-norms of the form $T = \varphi^{-1} \circ \text{Prod} \circ (\varphi \times \varphi)$, where $\text{Prod}(x,y) = x \cdot y$. This subclass is the *Product family*, $\mathcal{F}(\text{Prod})$.

Class (3) includes any t-norm that may be written as

$$T(x,y) = \begin{cases} a_i + (b_i - a_i) T_i\left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}\right) & (x,y) \in [a_i, b_i]^2 \\ \text{Min}(x,y) & \text{otherwise} \end{cases}$$

where $\{T_i, i \in J\}$ is a countable collection of archimedean t-norms and $(a_i, b_i) \subset [0,1]$ are disjoint intervals. Such continuous t-norms are called ordinal sums of archimedean t-norms or, for short, *ordinal sums*.

Obviously, $\{\text{Min}\} \cup \mathcal{F}(W) \cup \mathcal{F}(\text{Prod})$ is the set of all continuous t-norms that are not ordinal sums. Archimedean t-norms, that is, those that belong to $\mathcal{F}(W) \cup \mathcal{F}(\text{Prod})$, were characterized by the Aczél-Ling theorem: If T is a continuous binary operation on the unit interval, then T is an archimedean t-norm if and only if there exists a continuous and strictly decreasing function $f : [0,1] \to \mathbb{R}^+$ with $f(1) = 0$, called an *additive generator* of T, such that $T(x,y) = f^{(-1)}(f(x) + f(y))$ for any x, y in $[0,1]$, where $f^{(-1)}$ is the pseudo-inverse of f, defined by

$$f^{(-1)}(x) = \begin{cases} f^{-1}(x) & \text{if } x \in [0, f(0)] \\ 0 & \text{if } x \in (f(0), +\infty) \end{cases}$$

2 Indistinguishabilities as generalized distances

When I is a T-indistinguishability on X and $d = 1 - I$, the following characterization for T-indistinguishabilities was established in [4]:

Theorem 2.1 $I : X \times X \to [0,1]$ is a T-indistinguishability if and only if $d = 1 - I$ is a $\mathcal{T}^*$-generalized distance.

The following equivalence is interesting in order to upgrade the above result to families of t-norms:

Lemma 2.2 Let T be a t-norm and $T_\varphi \in \mathcal{F}(T)$. Then $I : X \times X \to [0,1]$ is a T_φ-indistinguishability if and only if $\varphi \circ I$ is a T-indistinguishability.

Proof. Trivial, since φ is an order automorphism. ■

This equivalence allows to easily extend the scope of theorem 2.1 to any t-norm expressed by means of an order automorphism:

Theorem 2.3 Let T be a t-norm and $T_\varphi \in \mathcal{F}(T)$. A function $I : X \times X \to [0,1]$ is a T_φ-indistinguishability if and only if $d = 1 - \varphi \circ I$ is a $\mathcal{T}^*$-generalized distance.

In the case of ordinary distances, last theorem's necessary condition remains valid for t-norms which are above Łukasiewicz t-norm:

Corollary 2.4 If $T_\varphi \in \mathcal{F}(T)$, T is a t-norm such that $T \geq W$ and $I : X \times X \to [0,1]$ is a T_φ-indistinguishability, then $d = 1 - \varphi \circ I$ is a distance on X with values in $[0,1]$.

Proof. I is a T_φ-indistinguishability, so $d = 1 - \varphi \circ I$ is a $\mathcal{T}^*$-generalized distance. In addition, $T \geq W$ if and only if $T^* \leq W^*$, and $W^* = \text{Min}(1, \text{Sum}) \leq \text{Sum}$, where Sum represents from now on the operation $\text{Sum}(x,y) = x + y$ in $\mathbb{R}$. ■ (see [1] for a different proof).

Last corollary may be applied, among others, to any t-norm $T_\varphi \in \mathcal{F}(T)$ such that T is both a t-norm and a 2-copula (see [3]), in particular to any $T_\varphi \in \mathcal{F}(\text{Prod}) \cup \mathcal{F}(W)$. Note also that when $T = \text{Min}$, then if I is a Min-indistinguishability, $d = 1 - \varphi \circ I$ is a distance for any order automorphism φ, since $\mathcal{F}(\text{Min}) = \{\text{Min}\}$.

Nevertheless, the reciprocal of corollary 2.4 is not valid for $T = \text{Min}$ nor for $T = \text{Prod}$ (it is true when $T = W$, as it will be shown in section 4, because $1 - id$ is an additive generator of W). Let us consider the distance on $\mathbb{R}$ defined as $d(x,y) = \frac{|x-y|}{1+|x-y|}$ for any $x, y \in \mathbb{R}$. Then values $x = 0$, $y = 1/2$ and $z = 1$ prove that $I = 1-d$ is not a Min-indistinguishability. For the product t-norm, it suffices to choose the euclidean distance on $X = [0,1]$, $d(x,y) = |x-y|$ for any x, y in $[0,1]$. In this case $I = \varphi^{-1} \circ (1-d)$ is not a T_φ-indistinguishability for any $T_\varphi \in \mathcal{F}(\text{Prod})$, and thus it is neither a Min-indistinguishability for any φ (take for example values $x = 0$, $y = 1/2$ and $z = 1$).

When corollary 2.4 is restricted to archimedean t-norms, the same result may be equivalently expressed by means of the t-norm's additive generator:

Corollary 2.5 Let T be an archimedean t-norm whose additive generator f verifies $f^{-1}(x) + f^{-1}(y) \leq 1 + f^{(-1)}(x+y)$ for any $x, y \in [0, f(0)]$. If $I : X \times X \to [0,1]$ is a T_φ-indistinguishability, $T_\varphi \in \mathcal{F}(T)$, then $d = 1 - \varphi \circ I$ is a distance on X with values in $[0,1]$.

Proof. The corollary's inequality is equivalent to $(1 - id) \circ f^{(-1)}(x+y) \leq (1-id) \circ f^{-1}(x) + (1-id) \circ f^{-1}(y)$, so $(1-id) \circ f^{(-1)}$ is sub-additive in $[0, f(0)]$. Then, as it is well known (see [3]), it is $W \leq T$, since $1 - id$ is an additive generator of W. As a consequence, applying corollary 2.4, $d = 1 - \varphi \circ I$ is a distance. ∎ (see [1] for a different proof).

Examples of functions that verify last corollary's condition are $f_1(x) = 1 - \sqrt{x}$ and $f_2(x) = (1-x)^2$.

3 More on distances and T_φ-indistinguishabilities

Let T be a continuous t-norm. Let $M_1(T, \text{Sum})$ be the set of functions $t : [0,1] \to \mathbb{R}^+ = [0, +\infty]$ which are continuous, strictly decreasing, and verify $t(1) = 0$ and $t \circ T \leq \text{Sum} \circ (t \times t)$. Let $M_2(T, \text{Sum})$ be the set of functions $s : \mathbb{R}^+ \to [0,1]$ which are continuous, strictly decreasing in $\mathbb{R}^+$ (or in an interval $[0, a]$, in which case it is $s(x) = 0$ for any $x \geq a$), verifying $s(0) = 1$ and $T \circ (s \times s) \leq s \circ \text{Sum}$. Obviously, the sets M_1 and M_2, introduced in [6] and reviewed in this paper's introduction, correspond, respectively, to $M_1(\text{Prod}, \text{Sum})$ and to the $M_2(\text{Prod}, \text{Sum})$ subset made of functions that are strictly decreasing in $\mathbb{R}^+$. In addition, if T_1 and T_2 are continuous t-norms such that $T_1 \leq T_2$, then $M_1(T_1, \text{Sum}) \subseteq M_1(T_2, \text{Sum})$, so $M_1(\text{Min}, \text{Sum})$ is the biggest set of this kind. In fact, it is $M_1(\text{Min}, \text{Sum}) = \{t : [0,1] \to \mathbb{R}^+; t \text{ continuous, strictly decreasing, } t(1) = 0\}$. On the contrary, if $T_1 \leq T_2$ then $M_2(T_2, \text{Sum}) \subseteq M_2(T_1, \text{Sum})$, being $M_2(\text{Min}, \text{Sum})$ the smallest one. It is even $M_2(\text{Min}, \text{Sum}) = \emptyset$, and the same happens for any ordinal sum, since the minimum operator is used to build this class of t-norms.

Next theorem shows how functions in $M_1(T, \text{Sum})$ allow to obtain distances from T_φ-indistinguishabilities.

Theorem 3.1 Let T be a continuous t-norm, $T_\varphi \in \mathcal{F}(T)$ and $t \in M_1(T, \text{Sum})$. If $I : X \times X \to [0,1]$ is a T_φ-indistinguishability on X, then $d = t \circ \varphi \circ I$ is a distance on X with values in $\mathbb{R}^+$.

Proof. It is easily proved using indistinguishabilities properties and $M_1(T, \text{Sum})$ definition. ∎

As $M_1 = M_1(\text{Prod}, \text{Sum})$, the method to construct distances from Prod-indistinguishabilities that was obtained in [6] is recovered with last theorem, taking $T = \text{Prod}$ and $\varphi = id$.

Examples of functions in $M_1(\text{Prod}, \text{Sum})$ are $t_0(x) = 1 - x$, $t_1(x) = -\log x$, $t_2(x) = \sqrt{-\log x}$ and $t_3(x) = \frac{\log x}{\log x - 1}$ (see [6]). Then, if I is a T_φ-indistinguishability on X, with $T_\varphi \in \mathcal{F}(\text{Prod})$, then $d_0 = 1 - \varphi \circ I$, $d_1 = -\log(\varphi \circ I)$, $d_2 = \sqrt{-\log(\varphi \circ I)}$ and $d_3 = \frac{\log(\varphi \circ I)}{\log(\varphi \circ I) - 1}$ are distances on X with values in $\mathbb{R}^+$.

Another example is given by the function $t(x) = 1 - x^2$, which belongs to $M_1(W, \text{Sum})$.

The reciprocal of theorem 3.1 is, in general, not true. It is neither true for $\mathcal{F}(\text{Prod})$ nor for $\mathcal{F}(W)$, and, as a consequence, nor for $T = \text{Min}$. For $\mathcal{F}(\text{Prod})$, it is sufficient to consider function $t(x) = 1 - x$ with the same example that was given for corollary 2.4. For $\mathcal{F}(W)$, if $t(x) = 1 - x^2$ and d is the euclidean distance on $[0,1]$, then $I = \varphi^{-1} \circ t^{(-1)} \circ d$ is not a T_φ-indistinguishability for any T_φ in $\mathcal{F}(W)$ (take for example the values $x = 0$, $y = 1/2$ and $z = 1$).

In a similar way as for last theorem, next result allows to obtain T_φ-indistinguishabilities from distances, but now functions in $M_2(T, \text{Sum})$ are used. Note that, as $M_2(T, \text{Sum})$ is empty both for $T = \text{Min}$ and for any ordinal sum, this result may only be applied to archimedean t-norms.

Theorem 3.2 Let T be an archimedean t-norm, $T_\varphi \in \mathcal{F}(T)$ and $s \in M_2(T, \text{Sum})$. If $d : X \times X \to \mathbb{R}^+$ is a distance, then $I = \varphi^{-1} \circ s \circ d$ is a T_φ-indistinguishability on X.

Proof. It is easily proved from distance's properties and $M_2(T, \text{Sum})$ definition. ∎

Some examples of T_φ-indistinguishabilities, with $T_\varphi \in \mathcal{F}(\text{Prod})$, are $I_0(x,y) = \varphi^{-1}\left(\frac{1}{1+d(x,y)}\right)$, $I_1(x,y) = \varphi^{-1}\left(e^{-d(x,y)}\right)$, $I_2(x,y) = \varphi^{-1} \circ \exp\left(-\sqrt{d(x,y)}\right)$ and

$I_3(x,y) = \varphi^{-1} \circ \exp\left(-\frac{d(x,y)}{d(x,y)+1}\right)$, since $s_0(x) = \frac{1}{1+x}$, $s_1(x) = e^{-x}$, $s_2(x) = e^{-\sqrt{x}}$ and $s_3(x) = e^{-\frac{x}{1+x}}$ belong to $M_2(\mathrm{Prod}, \mathrm{Sum})$ (see [6]). These examples are also T_φ-indistinguishabilities for $T_\varphi \in \mathcal{F}(W)$, since $M_2(\mathrm{Prod}, \mathrm{Sum}) \subset M_2(W, \mathrm{Sum})$.

The reciprocal of theorem 3.2 does not hold. To prove it, let us consider $I(x,y) = e^{-|x-y|}$, which is a Prod-indistinguishability on $\mathbb{R}$, and $s(x) = \frac{1}{1+x}$, which is a function belonging to $M_2(\mathrm{Prod}, \mathrm{Sum})$; function $d(x,y) = s^{-1} \circ I(x,y) = \frac{1-I(x,y)}{I(x,y)}$ is not a distance on $\mathbb{R}$: values $x = 0$, $y = 1$ and $z = 2$ allow to prove it.

4 Another characterization for T-indistinguishabilities

In the previous section, mechanisms to obtain distances from T_φ-indistinguishabilities and reciprocally were stated. The conjunction of theorems 3.1 and 3.2 trivially provides the following result, which characterizes T_φ-indistinguishabilities when T is an archimedean t-norm.

Theorem 4.1 Let T be an archimedean t-norm, $T_\varphi \in \mathcal{F}(T)$ and t a function in $M_1(T, \mathrm{Sum})$ such that $t^{(-1)} \in M_2(T, \mathrm{Sum})$. Then $I : X \times X \to [0,1]$ is a T_φ-indistinguishability on X if and only if $d = t \circ \varphi \circ I$ is a distance on X with values in $\mathbb{R}^+$.

In addition, it can be proved that function t in theorem 4.1 is an additive generator of the t-norm T:

Theorem 4.2 Let T be an archimedean t-norm. A function t in $M_1(T, \mathrm{Sum})$ is such that $t^{(-1)} \in M_2(T, \mathrm{Sum})$ if and only if t is an additive generator of T.

Proof. From $t \in M_1(T, \mathrm{Sum})$ and $t^{(-1)} \in M_2(T, \mathrm{Sum})$, since, in addition, it is $t^{(-1)} \circ t = Id_{[0,1]}$, it follows $T(x,y) = t^{(-1)} \circ t \circ T(x,y) \geq t^{(-1)}(t(x) + t(y)) \geq T(t^{(-1)} \circ t(x), t^{(-1)} \circ t(y)) = T(x,y)$, that is, t is an additive generator for T. Reciprocally, $t \in M_1(T, \mathrm{Sum})$ since $t \circ t^{(-1)}(x) \leq x$ holds for any $x \in \mathbb{R}^+$, and $t^{(-1)} \in M_2(T, \mathrm{Sum})$ is true because $t^{(-1)}[t \circ t^{(-1)}(x) + t \circ t^{(-1)}(y)] = t^{(-1)}(x+y)$. ∎

Last two theorems show that T_φ-indistinguishabilities, when $T_\varphi \in \mathcal{F}(T)$ and T is an archimedean t-norm, may be characterized by means of the t-norm's additive generator:

Theorem 4.3 Let T be an archimedean t-norm with additive generator f and $T_\varphi \in \mathcal{F}(T)$. Then $I : X \times X \to [0,1]$ is a T_φ-indistinguishability if and only if $d = f \circ \varphi \circ I$ is a distance on X with values in $\mathbb{R}^+$.

If $\varphi = Id$, this theorem summarizes the results previously obtained in [1] while directly working with additive generators. When particularized to the two subclasses of archimedean t-norms, it follows:

Theorem 4.4 If $T_\varphi \in \mathcal{F}(\mathrm{Prod})$, $I : X \times X \to [0,1]$ is a T_φ-indistinguishability if and only if $d(x,y) = -p \log(\varphi \circ I(x,y))$, $p \in (0, +\infty)$, is a distance in X. If $T_\varphi \in \mathcal{F}(W)$, $I : X \times X \to [0,1]$ is a T_φ-indistinguishability if and only if $d = 1 - \varphi \circ I$ is a distance on X.

Note that in the case $\varphi = Id$, Menger's Prod-indistinguishabilities theorem (see [2]) is obtained.

5 Conclusions

This paper provides a framework that relates ordinary distances with indistinguishability operators defined with respect to parametrized families of t-norms, $T_\varphi \in \mathcal{F}(T)$, where T is a continuous t-norm. In this context, distances may be obtained from T_φ-indistinguishabilities and reciprocally. In addition, when T is an archimedean t-norm, a characterization of T_φ-indistinguishabilities is given by means of the t-norm's additive generator.

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