ON THE STABILITY OF T-S FUZZY CONTROL FOR NON-LINEAR SYSTEMS

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Summary

The main feature of a Takagi-Sugeno (T-S) fuzzy model is to express the local dynamics of each fuzzy implication (rule) by a linear system model. The overall fuzzy model of the system is achieved by fuzzy "blending" of the linear system models. Parallel or feedback connections of T-S fuzzy systems which preserve the properties of each system are possible [1]. Thus a simple and straightforward approach for the control of non-linear systems has emerged [2]. By representing the non-linear system by a T-S type fuzzy model, linear feedback control techniques can be utilized to design a linear controller for each local linear model. The overall controller is a fuzzy blending of each individual linear controller and therefore non-linear but very simple to design. The closed loop fuzzy control system derived in this way is, in general, a system composed of rules with affine linear systems in their consequent parts. This design approach for the control of non-linear systems does not have the local character of a linear control design for the linearized, around an operating point, non-linear system, nor the complication and involvement of feedback linearization controllers. Its appeal is, however, dependent on the stability issues involved. While stability conditions have been exploited for T-S fuzzy systems composed of rules with linear consequent parts, the stability of T-S fuzzy systems composed of rules with affine linear system in their consequent parts needs further investigation.

Keywords: Fuzzy control, Non-linear systems, Stability

1 BACKGROUND MATERIAL

A continuous T-S fuzzy plant model is composed of n plant rules that can be represented as

Plant rule i

if x_1 is M_{1i} and x_2 is M_{2i} …and x_r is M_{ri}

then $\dot{x} = A_i x + B_i u$

where M_{pi} with $p = 1,..,r$ are fuzzy sets whose membership functions denoted by the same symbols are continuous piecewise polynomial functions and $x = \left[x_1 ....x_r\right]^T$ is the state vector.

Then, the final output of the T-S fuzzy system is inferred as follows:

$$\dot{x} = \sum_{i=1}^{n} w_i(x)(A_i x + B_i u) \qquad (1)$$

where the membership functions $w_i(x) = h_i(x) / \sum_{j=1}^{r} h_j(x)$, $h_i(x) = \prod_{j=1}^{r} M_{ji}(x_j)$ are non-negative and normalized. That is,

$$\sum_{i=1}^{n} w_i(x) = 1 .$$

We may also consider a T-S fuzzy control model composed of n rules having the same premises as those of the above plant, i.e.:

Controller rule i

if *x_1 is M_{1i} and x_2 is M_{2i} …and x_n is M_{ni}*

then $h = K_i x$

Then, the final output of the T-S fuzzy controller is inferred as follows:

$$h = \sum_{i=1}^{n} w_i(x) K_i x \qquad (2)$$

A closed loop control system can be constructed with a feedback connection of the two fuzzy blocks so that the control input of the plant is $u = r - h$ where r is a reference input. Then, the resulting closed loop control system is expressed by the fuzzy system:

System rule ij :

if *x is (M_{pi} and M_{pj})*

then $\dot{x} = (A_i - B_i K_j)x + B_i r \qquad (3)$

where x is $M_{pi} \Leftrightarrow x_1$ *is* M_{1i} *and… and* x_r *is* M_{ri}..
The membership function of the fuzzy set (M_{pi} *and* M_{pj}) is defined as $M_{pi} \times M_{pj}$ which is a continuous piecewise polynomial function (not necessarily convex) [1].
Thus, the final output of the closed loop T-S model is

$$\dot{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i(x)w_j(x)[(A_{ij}x + B_i r] \quad (4)$$

where $A_{ij} = A_i - B_i K_j$.
The stability of the free fuzzy system (1) (u=0) has been investigated in [1] using the Lyapunov direct method. The following stability theorem for the continuous time system holds.
Theorem 1: The equilibrium of the free fuzzy system

$$\dot{x} = \sum_{i=1}^{n} w_i(x) A_i x \quad (5)$$

is asymptotically stable in the large if there exist a common positive definite matrix $P = P^T > 0$ such that the Lyapunov inequality holds:

$$A_i^T P + PA_i < 0 \quad i=1,2,.., n \quad (6)$$

Theorem 1, can be used to derive the stability condition for system (4) when the reference input is zero(r=0), i.e. the stability condition for the closed loop system with zero reference:

$$\dot{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i(x)w_j(x)A_{ij}x \quad (7)$$

is to find a common matrix $P = P^T > 0$ such that the following Lyapunov inequality holds:

$$A_{ij}^T P + PA_{ij} < 0 \quad i,j=1,2,.., n \quad (8)$$

We can write (7) as [1]:

$$\dot{x} = \sum_{i=1}^{n} w_i(x)w_i(x)[(A_{ii}x + 2\sum_{i<j} w_i(x)w_j(x)G_{ij}x] \quad (9)$$

where $G_{ij} = \dfrac{A_{ij} + A_{ji}}{2}, i < j$

Thus, we can state the stability for the closed loop system in the following theorem [2].
Theorem 2: The equilibrium of the fuzzy system (7) or (9) is asymptotically stable in the large if there exist a common positive definite matrix $P=P^T>0$ such that the following inequalities hold:

$$A_{ii}^T P + PA_{ii} < 0 \quad G_{ij}^T P + PG_{ij} < 0 \quad (10)$$

Stability theorems 1 and 2 involve the solution of matrix inequalities and are solved numerically by using LMI convex programming techniques.
A special case:
An analytical stability solution exists in the following special case: If (i) (A_i, B_i) are controllable pairs (ii) $B_i=B$ and (iii) we can choose K_i such that:

$$A = A_i - BK_i \quad i=1, ., n \quad (11)$$

with A Hurwitz then the system is stable. Note than in this case $G_{ij} = A$ and we can therefore choose P such that $A^T P + PA < 0$. However, a choice of K_i such that $A = A_i - BK_i$ may not be possible even if (A_i, B_i) is controllable.

We have seen that stability conditions have been derived for T-S fuzzy free systems like (5), (7) i.e. systems composed of rules with linear consequent parts. In general, however, the bias component in (3) may be either a non-zero constant (for e.g. in a regulation problem *r=const*) or time varying (in trajectory following *r(t)*). Furthermore, fuzzy modelling of a non-linear system may result in fuzzy systems composed of rules with non-zero bias in their consequent parts. For e.g., we can consider a T-S fuzzy model of a system composed of n rules that can be represented as:
System rule i
if x_1 *is* M_{1i} *and* x_2 *is* M_{2i} *…and* x_n *is* M_{ni}
then $\dot{x} = A_i x + d_i$ (12)
Therefore, it is interesting to study the stability of T-S fuzzy systems in the form of (12), which we subsequently call fuzzy affine systems.

2 STABILITY OF FUZZY AFFINE LINEAR SYSTEMS

Let us consider a T-S fuzzy system model composed of n rules that can be represented by (12). The final output of the T-S fuzzy system is inferred as :

$$\dot{x} = \sum_{i=1}^{n} w_i(x)[A_i x + d_i] \quad (13)$$

Let us assume that the equilibrium point of this system is x=0. That is, we assume that:

$$\sum_{i=1}^{n} w_i(0)d_i = 0 \quad (14)$$

We can express this system as a T-S fuzzy system model composed of n rules that can be represented as:
System rule i
if x_1 *is* M_{1i} *and* x_2 *is* M_{2i} *…and* x_n *is* M_{ni}
then $\dot{\bar{x}} = \bar{A}_i \bar{x}$

where $\bar{A}_i = \begin{bmatrix} A_i & d_i \\ 0 & 0 \end{bmatrix}$, $\bar{x} = \begin{bmatrix} x & 1 \end{bmatrix}^T$

and thus, the final output of the T-S fuzzy system is inferred as : $\dot{\bar{x}} = \sum_{i=1}^{n} w_i(x)\bar{A}_i \bar{x}$ (15)

We can state the stability of (15) as follows.
Theorem 3: The equilibrium x=0 of the affine fuzzy system is asymptotically stable in the large if there exist a

common positive definite matrix $\overline{P} = \overline{P}^T > 0$ such that the following Lyapunov inequality holds:

$$\overline{A_i}^T \overline{P} + \overline{P}\overline{A_i} < 0 \quad \text{i=1,2,.., n} \tag{16}$$

It is interesting to note that the stability of the fuzzy free system, although required, it is not sufficient to guarantee the stability of the affine fuzzy system. That is, the existence of a common positive definite matrix $P = P^T > 0$ such that

$$A_i^T P + PA_i < 0 \quad \text{i=1,2,.., n}$$

does not imply the existence of a common $\overline{P}$. It would be useful however to relate the stability of the affine fuzzy system with the stability of the corresponding free system and thus, investigate the extra conditions which are required for ensuring the stability of the equilibrium point and the parameters which influence stability. We will do so with the help of a simple example.

3 EXAMPLE: SINGLE-LINK ROBOT ARM

We demonstrate the fuzzy control design for the regulation of a single link robot arm (driven pendulum). The arm is governed by the equation:

$$\ddot{\theta} + a\dot{\theta} + b\sin\theta = u, \quad a, b > 0 \tag{17}$$

where $\theta = 0$ corresponds to the lower vertical position, u is the normalised torque input, the a term corresponds to viscous damping while b depends on the gravity and the distribution of mass.
As a first step in the design procedure, we must represent the non-linear system by a T-S fuzzy model.
The above system may be approximated by a fuzzy blend of linear systems in the form of (1).

$$A_i = \begin{bmatrix} 0 & 1 \\ -b\sin\theta_{oi} / \theta_{oi} & -a \end{bmatrix}, \ \theta_{oi} = 0, \pm\pi/2$$

$$B = \begin{bmatrix} 0 & 1 \end{bmatrix}^T \ \text{and} \ x = \begin{bmatrix} \theta & \dot{\theta} \end{bmatrix}^T. \tag{18}$$

Specifically

$$A_1 = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ -2b/\pi & -a \end{bmatrix}, \ A_3 = \begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix}$$

The fuzzy sets are:

$$w_1(\theta) = \begin{cases} 1 - 2/\pi\, |\theta|, \theta \in D_1 \\ 0, \theta \in D_2 \end{cases}, \tag{19}$$

$$w_3(\theta) = \begin{cases} 0, \theta \in D_1 \\ 2/\pi\, |\theta| - 1, \theta \in D_2 \end{cases}$$

and $w_2(\theta) = 1 - w_1 - w_2$, where $D_1 = [-\pi/2, \pi/2]$, $D_2 = [\pi/2, \pi] \cup [-\pi, -\pi/2]$.
The T-S fuzzy model which approximates the non-linear system is given by the inferred system:

$$\dot{x} = \sum_{i=1}^{n} w_i(\theta)(A_i x + Bu) \tag{20}$$

The next step is the design of a linear controller for each local linear model. The control objective is to reach a desired angle θ_d so that

$$x_d = [\theta_d \quad 0]^T.$$

(i) If the desired angle θ_d is zero, the fuzzy control input is defined as

$$u = 0 - \sum_{i=1}^{n} w_i(\theta) K_i x$$

and the closed loop system is a free fuzzy system which further belongs to the special case discussed in section 2. Thus, it can be reduced to a LTI system. Specifically, since $B = B_i$ we can define a stability matrix A expressing the desired dynamics of the arm behaviour and try to find feedback gains K_i satisfying matrix equation (11).
Let,

$$A = \begin{bmatrix} 0 & 1 \\ -c & -d \end{bmatrix}, \ c, d > 0$$

and K_i are calculated to satisfy $A = A_i - BK_i$.
Then the T-S fuzzy model of the closed loop system becomes:

$$\dot{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i(\theta) w_j(\theta)(A_i - BK_i)x \Leftrightarrow \dot{x} = Ax$$

(ii) When the desired angle is non zero, we can transform the equilibrium point of the non-linear system to zero, by providing the torque required at steady state, $u_s = b\sin\theta_d$. Thus, set $u = u_s + u'$, and define the state variables as $e = \begin{bmatrix} \theta - \theta_d & \dot{\theta} \end{bmatrix}^T$ so that we can write the T-S fuzzy plant model as:

$$\dot{e} = \sum_{i=1}^{n} w_i(\theta)(A_i e + Bu' + [A_i x_d + Bu_s]) \tag{21}$$

where the quantity into brackets is a constant bias and therefore we have the case of a system modelled as a fuzzy blend of affine linear systems.
It is easy to show that the free system equilibrium is zero, i.e. (e=0 or x=x_d for u'=0):

$$0 = \sum_{i=1}^{n} w_i(\theta_d)(A_i x_d + Bu_s) \tag{22}$$

If we design the fuzzy controller as before and set:

$$u' = -\sum_{i=1}^{n} w_i(x) K_i e$$

the T-S fuzzy model of the closed loop system is:

$$\dot{e} = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i(\theta) w_j(\theta)\{(A_i - BK_i) + $$

$$+ \sum_{i=1}^{n} w_i(\theta)(A_i x_d + Bu_s)$$

or $$\dot{e} = Ae + \sum_{i=1}^{n} w_i(\theta)(A_i x_d + Bu_s) \qquad (23)$$

For asymptotic stability of this system at x_d it is necessary that the linearization of (23) yields a stable system. Locally (23) can be described using (22) by the following linear system:

$$\dot{e} = A_{x_d} e \qquad (24)$$

$$A_{x_d} = A + \sum_i A_i x_d \frac{\partial w_i}{\partial x}(x_d)$$

In the example, $A_i x_d = \begin{bmatrix} 0 \\ -b\dfrac{\sin\theta_{o,i}}{\theta_{o,i}}\theta_d \end{bmatrix}$ and after some algebra we obtain that

$$A_{x_d} = \begin{cases} A + \begin{bmatrix} 0 & 0 \\ b\dfrac{2}{\pi}\left(1 - \dfrac{2}{\pi}\right)|\theta_d| & 0 \end{bmatrix}, & \theta_d \in D_1 \\ A + \begin{bmatrix} 0 & 0 \\ b\left(\dfrac{2}{\pi}\right)^2 |\theta_d| & 0 \end{bmatrix}, & \theta_d \in D_2 \end{cases} \qquad (25)$$

Note that the derivative in (24) is defined everywhere except $\pm\pi/2$.

For $-c + b\dfrac{2}{\pi}\left(1 - \dfrac{2}{\pi}\right)|\theta_d| > 0$, $\theta_d \in D_1$

and $-c + b\left(\dfrac{2}{\pi}\right)^2 |\theta_d| > 0$, $\theta_d \in D_2$ (26)

the system is locally unstable. To ensure local stability in this example c had to be large enough to dominate in the above terms. This can be achieved with faster closed loop dynamics.

Therefore, a stable A (c, d>0) is not sufficient for the system stability. For global stability further analysis is required.

4 LYAPUNOV STABILITY CONDITIONS FOR THE SIMPLIFIED SYSTEM

Consider the closed loop equation of a simplified plant like (23).

$$\dot{e} = Ae + \sum_{i=1}^{n} w_i(x)(A_i x_d + Bu_s) \qquad (27)$$

We may rewrite this system using the equation of the equilibrium condition (22) as:

$$\dot{e} = Ae + \sum_{i=1}^{n} A_i x_d[(w_i(x) - w_i(x_d)]$$

$$\dot{e} = Ae + C_d \Delta_d(x) \qquad (28)$$

where C_d is a matrix and Δ_d a vector defined as:

$C_d = [A_1 x_d \ldots A_p x_d]$, $\Delta_d = [w_i(x) - w_i(x_d)]_i$.

Let the Lyapunov candidate function be

$$V(e) = e^T Pe \text{ , } P > 0 \text{ .}$$

Differentiate over time along system trajectories to obtain:

$$\dot{V}(e) = -e^T Qe + 2e^T PC_d \Delta_d(x) \qquad (29)$$

where Q is the positive definite matrix given by:

$$A^T P + PA + Q = 0 \qquad (30)$$

Now assuming $w_i(x)$ are Lipschitz and since $\Delta_d(x_d) = 0$, there exists a positive γ such that

$$\|C_d \Delta_d(e + x_d)\| \le \gamma\|e\| \qquad (31)$$

for $e \in D$: the domain of Δ_d provided that D is a compact set. Then (30) holds for the whole of the state-space.

$$\begin{aligned} \dot{V}(e) &= -e^T Qe + 2e^T PC_d \Delta_d(x) \\ &\le -e^T Qe + 2\gamma\|P\|\|e\|^2 \\ &= -e^T(Q - 2\gamma\|P\|I)e \end{aligned}$$

Theorem 4: The equilibrium x=x_d of the affine fuzzy system (27) is asymptotically stable in the large if there exist a positive γ such that (31) is satisfied and

$$(\lambda_{\min}(Q) - 2\gamma\|P\|I) > 0 \text{ .} \qquad (32)$$

where Q and P are defined in (30).

Note that this condition depends on γ which depends on $w_i(x)$. In the single link robot example γ is evaluated to be $\gamma = 4/\pi$ obtained by a two dimensional search for θ and θ_d .

5 CONCLUSIONS

Stability issues regarding the T-S fuzzy control of non-linear systems have been considered and illustrated with the use of the single link robot regulation problem. Past work on stability of fuzzy systems can be easily extended when the aim is to stabilize the system at zero reference. When a non-zero set-point is present, stability conditions of the free system can not be carried through. Extra conditions are needed to guarantee stability. These conditions depend on the choice of the particular fuzzy set memberships. Future work includes the stability study for the general case with constant and time varying bias. Also of concern are robustness issues under modelling errors and parameter uncertainties.

References

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