The Class of Max-C Projection Autoassociative Fuzzy Memories

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Abstract. A fuzzy associative memory (FAM) is an input-output system that allows for the storage and recall of fuzzy sets. In this paper, we present and discuss a new class of autoassociative FAM models, called max-C projection autoassociative fuzzy memories (max-C PAFMs), which have been derived from autoassociative fuzzy morphological memories (AFMMs). In few words, a max-C PAFM projects the input fuzzy set into the set of the max-C combinations of the stored items. We illustrate and compare the performance of some max-C PAFM models with other AFMMs for the storage and recall of corrupted scale-gray images.

Keywords: associative memory, mathematical morphology, adjunction, fuzzy relational inequalities, image reconstruction.

1 Introduction

Associative memories (AMs) are mathematical models inspired by the human brain ability to store and retrieve a determined information by association [1]. An AM is categorized as either heteroassociative or autoassociative. An autoassociative memory is designed for the storage of a finite set $A = \{a_1, \ldots, a_k\}$. The famous Hopfield neural network is an example of an autoassociative memory [2].

We speak of a fuzzy associative memory (FAM) if the AM is designed for the storage and recall of fuzzy sets, that is, $A \subseteq \mathbb{[0,1]}^n$ [3, 4]. Applications of fuzzy associative memories include control [3], classification [5], time series prediction [6, 7, 8], and restoration of corrupted images [7, 9]. The reader interested on a comprehensive review on fuzzy associative memories is invited to consult [4]. In this paper, we only address autoassociative fuzzy memories. Precisely, we focus on the class of autoassociative fuzzy morphological memories (AFMMs) proposed by Valle and Sussner [7, 10].

The AFMMs can be seen as fuzzy versions of the autoassociative morphological memories (AMMs) introduced by Ritter and Sussner [11]. Like the AMM models, AFMMs exhibit unlimited absolute storage capacity and an excellent tolerance to either erosive or dilative noise. Despite the successful applications...
of morphological and fuzzy morphological autoassociative memories [6, 7, 8, 12], these memories suffer from a large number of spurious memories.

Recently, Valle introduced the max-plus projection autoassociative morphological memory (max-plus PAMM), which have less spurious memories than the original AMM [13]. Inspired by the max-plus PAMM, we proposed the new class of max-\(C\) projection autoassociative fuzzy memories (max-\(C\) PAFMs) [14]. In general terms, a max-\(C\) PAFM projects the input fuzzy set into the set whose elements are all max-\(C\) combinations of the stored items. Such as the max-plus PAMMs, the max-\(C\) PAFMs exhibit unlimited absolute storage capacity and an excellent tolerance to dilative noise. On the downside, they are very sensitive to either erosive or mixed noise.

The paper is organized as follows. The next section briefly reviews the min-\(D\) AFMMs. The max-\(C\) PAFMs are introduced in Section 3. Computational experiments comparing the performance of the new models and min-\(D\) AFMMs for the storage and recall of scale-gray images are given in Section 4. The paper finishes with the concluding remarks in Section 5. We would like to point out that this paper corresponds to an improved version of the conference paper [14]. Namely, in this paper we provide some new theoretical results and we provide further computational experiments.

2 Autoassociative Fuzzy Morphological Memories

In this section, we briefly review the autoassociative fuzzy morphological memories (AFMM). The reader interested on a detailed account on this subject is invited to consult [7, 10]. Let us begin by recalling some basic concepts from fuzzy set theory and fuzzy logic.

Throughout the paper, we only consider fuzzy sets on a finite universe of discourse. A fuzzy set \(x\) on a finite universe \(U = \{u_1, u_2, \ldots, u_n\}\) corresponds to a vector \(x = [x_1, x_2, \ldots, x_n]^T \in [0, 1]^n\), where the component \(x_j = x(u_j)\) denotes the degree of membership of \(u_j\) on the fuzzy set \(x\).

A fuzzy disjunction \(D\) is an increasing mapping \(D : [0, 1] \times [0, 1] \rightarrow [0, 1]\) such that \(D(0, 0) = 0\) and \(D(1, 0) = D(0, 1) = 1\). An hybrid monotonous (decreasing in the first argument and increasing in the second argument) operator \(J : [0, 1] \times [0, 1] \rightarrow [0, 1]\) is a fuzzy co-implication if it satisfies \(J(0, 0) = J(1, 1) = 0\) and \(J(0, 1) = 1\) [15]. We say that a commutative fuzzy disjunction \(D\) and a fuzzy co-implication \(J\) form an adjunction if and only if the following equivalence holds for \(x, y, z \in [0, 1]\):

\[
D(x, y) \geq z \iff x \geq J(y, z). \tag{1}
\]

From now on, we assume that a fuzzy disjunction \(D\) is commutative and forms an adjunction with a certain fuzzy co-implication \(J\).

An autoassociative fuzzy memory is designed for the storage of a family of fuzzy sets \(A = \{a^1, \ldots, a^k\} \subseteq [0, 1]^n\), called fundamental memories set. In mathematical terms, an autoassociative fuzzy memory is an application \(M : [0, 1]^n \rightarrow [0, 1]^n\) such that the identity \(M(a^k) = a^k\) holds as far as possible for
Let $\xi \in K = \{1, 2, \ldots, k\}$. Moreover, $M$ should exhibit some noise tolerance, i.e., $M(\tilde{a}\xi) = a\xi$ is expected to hold true for noisy or incomplete versions $\tilde{a}\xi$ of $a\xi$.

Motivated by concepts from mathematical morphology, Valle and Sussner introduced the autoassociative fuzzy morphological memories (AFMMs) [4, 10]. Due to page limit, in this paper we focus only on the min-$D$ AFMM.

Let $D$ be a commutative fuzzy disjunction and $J$ a fuzzy co-implication such that $D$ and $J$ forms an adjunction. Given a fundamental memory set $A = \{a^1, \ldots, a^k\} \subseteq [0, 1]^n$, a min-$D$ AFMM $M : [0, 1]^n \rightarrow [0, 1]^n$ is defined by
\[ M(x) = M \bullet x, \]
where the symbol $\bullet$ denotes a min-$D$ product based on the fuzzy disjunction $D$.

The matrix $M \in [0, 1]^{n \times n}$ is called the synaptic weight matrix of the AFMM. We would like to point out that $M$ given by (2) corresponds to an erosion of fuzzy mathematical morphology [16, 17]. Hence the name fuzzy morphological memory. A new theoretical justification for a subclass of AFMMs has been proposed recently byPerfilieva et al. using fuzzy preorder [18].

The synaptic weight matrix $M$ can be determined using the fuzzy learning by adjunction [10]. Let $A = [a^1, \ldots, a^k]$ be the matrix whose columns correspond to the fundamental memories. The fuzzy learning by adjunction establishes that the matrix $M$ is the best approximation from above of the matrix $A$ in terms of the min-$D$ product. Formally, the fuzzy learning by adjunction defines
\[ M = \bigwedge \{ V \in [0, 1]^{n \times n} : V \bullet A \geq A \}. \]
Alternatively, the solution of (3) can be expressed using the equation
\[ M = A \triangleright A^T, \]
where the symbol $\triangleright$ denotes the max-$J$ product based on the fuzzy co-implication $J$ which forms an adjunction with the fuzzy disjunction $D$ [10].

The following proposition reveals that a min-$D$ AFMM $M$ projects the input $x$ into the set of all fixed points of the memory [7]. Alternatively, the output $M(x)$ is the largest fixed point less than or equal to the input $x$.

**Proposition 1** If $D$ is a commutative and associative fuzzy disjunction with an identity then, for any input $x \in [0, 1]^n$, the output of a min-$D$ AFMM $M$ satisfies
\[ M(x) = \bigvee \{ z \in I(A) : z \leq x \}, \]
where $I(A) = \{ z \in [0, 1]^n : M(z) = z \}$ denotes the set of all fixed points of $M$, which depends on the fundamental memory set $A = \{a^1, \ldots, a^k\}$.

In the light of Proposition 1, from now on we assume that the fuzzy disjunction $D$ is commutative, associative, and has an identity. In this case, it is not hard to show that $A \subseteq I(A)$, i.e., all fundamental memories are fixed points of the min-$D$ AFMM. In other words, the identity $M(a^\xi) = a^\xi$ holds true for any
ξ ∈ {1, . . . , k}. Also, we can show that several transformations of the fundamental memories as well as any minimax combinations of these transformations are fixed points of \( \mathcal{M} \) [7]. Now, since an element on the set difference \( \mathcal{I}(A) \setminus A \) is a spurious memory of \( \mathcal{M} \), we deduce that a min-D AFMM has a large number of spurious memories. Moreover, we conclude from (5) that the identity \( \mathcal{M}(x) = a^\xi \) is satisfied only if \( x \geq a^\xi \). In other words, a min-D AFMM exhibits tolerance with respect to dilative noise, but it is extremely sensitive to either erosive or mixed (= dilative + erosive) noise. We would like to recall that \( x \) corresponds to a dilated version of a fundamental memory \( a^\xi \) if \( x \geq a^\xi \). Dually, \( x \) is an eroded version of the fundamental memory \( a^\xi \) if \( x \leq a^\xi \) [11].

**Example 1** Consider the fundamental memory set

\[
A = \left\{ a^1 = \begin{bmatrix} 0.6 \\ 0.5 \\ 0.1 \\ 0.8 \end{bmatrix}, a^2 = \begin{bmatrix} 0.9 \\ 0.3 \\ 0.5 \\ 0.2 \end{bmatrix}, a^3 = \begin{bmatrix} 0.2 \\ 0.5 \\ 0.6 \\ 0.8 \end{bmatrix} \right\}
\]

and the fuzzy disjunction \( D_M(x, y) = x \lor y \). Note that \( D_M \) is commutative, associative, and has an identity. Furthermore, \( J_M(x, y) = \begin{cases} 0, & x \geq y, \\ y, & x < y, \end{cases} \) is the fuzzy co-implication that forms an adjunction with \( D_M \). From (4), we obtain the matrix

\[
M_M = A \, \overset{\bullet}{\otimes}_M \, A^T = \begin{bmatrix} 0 & 0 & 0.9 & 0.9 \\ 0.5 & 0.5 & 0.3 \\ 0.6 & 0.6 & 0 & 0.5 \\ 0.8 & 0.8 & 0.8 & 0 \end{bmatrix},
\]

where \( \overset{\bullet}{\otimes}_M \) is the max-J product based on the fuzzy co-implication \( J_M \). Now, given the input vector

\[
x = \begin{bmatrix} 0.7 \\ 0.6 \\ 0.1 \\ 0.9 \end{bmatrix}^T,
\]

the min-D \( M \) AFMM yields

\[
M_M(x) = M \, \overset{\bullet}{\otimes}_M \, x = \begin{bmatrix} 0.7 \\ 0.5 \\ 0.1 \\ 0.8 \end{bmatrix}^T \neq a^1,
\]

where \( \overset{\bullet}{\otimes}_M \) is the min-D product based on the fuzzy disjunction \( D_M \). Note that the input \( x \) corresponds to a dilated version of the fundamental memory \( a^1 \). In fact, we have \( x = a^1 + [0.1 \ 0.1 \ 0.1]^T \geq a^1 \). In this example, however, the min-D \( M \) AFMM failed to produce the desired output \( a^1 \). Also, since \( M(x) \) does not belong to the fundamental memory set \( A \), the fuzzy set \( [0.7, 0.5, 0.1, 0.8]^T \) is a spurious memory.

### 3 Max-C Projection Autoassociative Fuzzy Memories

First of all, an increasing mapping \( C : [0, 1] \times [0, 1] \to [0, 1] \) is a fuzzy conjunction if \( C(1, 0) = C(0, 1) = 0 \) and \( C(1, 1) = 1 \). Given a fuzzy conjunction \( C \), the set
of all max-C combinations of vectors from \( A = \{a^1, \ldots, a^k\} \subseteq [0, 1]^n \) is

\[
C(A) = \left\{ \bigvee_{\xi=1}^k C(\lambda_{\xi}, x^\xi) : \lambda_{\xi} \in [0, 1] \right\}.
\] (10)

Note that \( z \in C(A) \) if and only if \( z_i = \bigvee_{\xi=1}^k C(\lambda_{\xi}, a^\xi_i) \) for all \( i = 1, \ldots, n \).

A fuzzy implication \( I : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is defined as a hybrid monotonous mapping such that \( I(0, 0) = I(1, 1) = 1 \) and \( I(1, 0) = 0 \) [15]. A fuzzy implication \( I \) and a fuzzy conjunction \( C \) form an adjunction if and only if

\[
C(x, y) \leq z \iff x \leq I(y, z), \quad \forall x, y, z \in [0, 1].
\] (11)

Similarly, in this paper we assume that \( C \) is a fuzzy conjunction that form an adjunction with a certain fuzzy implication \( I \).

A max-C projection autoassociative fuzzy memory (max-C PAFM), introduced recently by us in [14], is obtained by replacing \( I(A) \) by the set of all max-C combinations of the fundamental memories in (5). Formally, given a fundamental memory set \( A \), a max-C PAFM \( V : [0, 1]^n \rightarrow [0, 1]^n \) is defined by

\[
V(x) = \bigvee \{ z \in C(A) : z \leq x \}, \quad \forall x \in [0, 1]^n.
\] (12)

In words, a max-C PAFM projects the input \( x \) into the set of all max-C combination of the fundamental memories. Therefore, like a min-D AFMM, a max-C PAFM exhibits a certain tolerance to dilative noise but it is extremely sensitive to either erosive or mixed noise. Furthermore, the following theorem provides an effective formula for the implementation of a max-C PAFM [14]:

**Theorem 1** Let \( C \) be a fuzzy conjunction that forms an adjunction with a fuzzy implication \( I \). Given a fundamental memory set \( A = \{a^1, \ldots, a^k\} \), a max-C PAFM satisfies the following for any input \( x = [x_1, \ldots, x_n]^T \in [0, 1]^n \):

\[
V(x) = \bigvee_{\xi=1}^k C(\lambda_{\xi}, a^\xi), \text{ where } \lambda_{\xi} = \bigwedge_{j=1}^n I(a^\xi_j, x_j), \quad \forall \xi \in \{1, \ldots, k\}.
\] (13)

Moreover, \( V(x) \leq x \) and \( V(V(x)) = V(x) \) holds for all \( x \in [0, 1]^n \).

**Remark 1** The parameter \( \lambda_{\xi} \) in (13) measures, in the sense of Bandler-Kohout, the degree of inclusion of the fundamental memory \( a^\xi \) in the input \( x \) [17].

The following theorem says that, like the min-D AFMM, a max-C PAFM exhibit optimal absolute storage capacity if the fuzzy conjunction \( C \) has a left identity.

**Theorem 2** If \( C \) is a fuzzy conjunction with an identity, the max-C PAFMs satisfies \( V(a^\xi) = a^\xi \) for any fundamental memory set \( A = \{a^1, \ldots, a^k\} \).
Example 2  Consider the fundamental memory set $A$ given by (6). Let $C_M$ and $I_M$ be the minimum fuzzy conjunction and Gödel fuzzy implication defined respectively by $C_M(x,y) = x \land y$ and $I_M(x,y) = \begin{cases} 1, & x \leq y, \\ y, & x > y. \end{cases}$ Note that $C_M$ is a fuzzy conjunction with 1 as identity. Moreover, $C_M$ and $I_M$ form an adjunction. We synthesized the max-$C_M$ PAFM designed for the storage of $A$ using the adjunction pair $(I_M,C_M)$. We first confirmed that the equation $V_M(a^\xi) = a^\xi$ holds for $\xi = 1, 2, 3$. Then, we presented the vector $x$ defined by (8) as input to the max-$C_M$ PAFM. We obtained from (13) the following coefficients:

$$\lambda_1 = 1.0, \quad \lambda_2 = 0.1, \quad \text{and} \quad \lambda_3 = 0.1.$$  

Hence, the output of the max-$C_M$ PAFM is

$$V_M(x) = C_M(\lambda_1,a^1) \lor C_M(\lambda_2,a^2) \lor C_M(\lambda_3,a^3) = \begin{bmatrix} 0.6 \\ 0.5 \\ 0.1 \\ 0.8 \end{bmatrix} \lor \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix} \lor \begin{bmatrix} 0.6 \\ 0.5 \\ 0.1 \\ 0.8 \end{bmatrix} = a^1.$$  

Different from the min-$D_M$ AFMM, the max-$C_M$ PAFM retrieved the fundamental memory $a^1$. Moreover, this example shows that max-$C_M$ PAFMs can be more robust to dilative noise than its corresponding min-$D_M$ AFMM.

Let us conclude the section by emphasizing that we cannot ensure optimal absolute storage capacity if $C$ does not have an identity.

Example 3  Consider the “compensatory and” fuzzy conjunction defined by

$$C_Z(x,y) = \sqrt{(x+y-xy)}.$$  

Note that $C_z$ does not have a left identity. Moreover, the fuzzy implication that forms an adjunction with $C_z$ is

$$I_Z(x,y) = \begin{cases} 1, & x = 0, \\ 1 \land \left[ \frac{-x^2 + \sqrt{x^2 + 4x(1-x)y^2}}{2x(1-x)} \right], & 0 < x < 1, \\ y^2, & x = 1. \end{cases}$$  

Now, let $V_Z : [0,1]^4 \rightarrow [0,1]$ be the max-$C_Z$ PAFM designed for the storage of the fundamental memory set $A$ given by (6). Upon presentation of the fundamental memory $a^1$ as input we have from (13) the following coefficients:

$$\lambda_1 = 0.28, \quad \lambda_2 = 0.04, \quad \text{and} \quad \lambda_3 = 0.03.$$  

Thus, the output of the max-$C_Z$ PAFM $V_Z$ is

$$V_Z(a^1) = C_Z(\lambda_1,a^1) \lor C_Z(\lambda_2,a^2) \lor C_Z(\lambda_3,a^3) = \begin{bmatrix} 0.35 \\ 0.30 \\ 0.10 \\ 0.43 \end{bmatrix} \neq a^1.$$
In a similar fashion, we obtain from the fundamental memories $a^2$, and $a^3$, the fuzzy sets

$$V_Z(a^2) = \begin{bmatrix} 0.57 \\ 0.26 \\ 0.37 \\ 0.2 \end{bmatrix} \neq a^2 \quad \text{and} \quad V_Z(a^3) = \begin{bmatrix} 0.20 \\ 0.37 \\ 0.42 \\ 0.52 \end{bmatrix} \neq a^3. \quad (19)$$

Note that, in accordance with Theorem 1, the inequality $V_Z(a^\xi) \leq a^\xi$ holds for $\xi = 1, 2, 3$. On the downside, the max-$C_Z$ PAFM failed to yield the desired responses $a^1$, $a^2$, and $a^3$.

4 Computational Experiments

Let us compare the performance of max-$C$ PAFMs and min-$D$ AFMMs for the storage and recall of scale-gray images. Precisely, consider the twelve gray-scale images of size $64 \times 64$ shown in Fig. 1. These images have been identified with fuzzy sets $a^\xi \in [0, 1]^{4096}$, for all $\xi = 1, 2, \ldots, 12$. The fundamental memory set $A = \{a^1, \ldots, a^{12}\} \subseteq [0, 1]^{4096}$ was stored in the min-$D$ AFMMs $M_M$, $M_P$, and $M_L$, which are obtained using respectively the maximum, the probabilistic sum, and the Lukasiewicz fuzzy disjunction [7]. We also used the fundamental memory set $A$ to synthesize the max-$C$ PAFMs $V_M$, $V_P$, and $V_L$, obtained using respectively the minimum, the product, and the Lukasiewicz fuzzy conjunction.

For comparison purposes, we also stored the same fundamental memory set into the optimal linear associative memory (OLAM) [19], the kernel associative memory (KAM) [20], and the recurrent exponential fuzzy associative memory (RE-FAM) [9].

We first confirmed that the nine AMs exhibit optimal absolute storage capacity. Then, we fed them with dilated versions of the fundamental memories. Precisely, in the first scenario, the original images have been corrupted by adding

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The experiments were conducted on MATLAB in a computer with processor Intel Core i7-5500U, 2.50GHZ, and 8GB RAM.
the absolute value of a Gaussian noise with zero mean and variance $\sigma^2$ between 0.02 and 0.9. In the second scenario, we introduced salt noise with probability $\rho$ ranging from 0.02 to 0.9. The average peak signal-to-noise ratio (PSNR) obtained from 360 trials, i.e., each original image was corrupted 30 times for a certain noise intensity, is shown in Figures 2 and 3. We would like to point out that, in order to avoid infinities, we bounded the PSNR rates by 100.

Note that the new memories $V_M, V_P$ and $V_L$ outperformed the models $M_L, M_P, M_M$, OLAM, and KAM. For large intensities of noise, the new models yielded PSNR rates larger than the RE-FAM. Furthermore, the min-$D_M$ AFMM $M_M$ is the worst min-$D$ AFMM. In contrast, the max-$C_M$ PAFM $V_M$ yielded excellent PSNR rates in both scenarios.

Finally, Fig. 4 provides a visual interpretation of the error correction capability of the min-$D_L$ AFMM and the max-$C_M$ PAFM – the best min-$D$ AFMM
Fig. 4. Error correction capability of the autoassociative memories.

and the best max-$C$ PAFM. This figure also shows the error correction capability of the RE-FAM, KAM, and OLAM models. Specifically, the first row exhibits dilated versions of the fundamental memories obtained by: a) introducing salt noise with probability $\rho = 0.4$. b) adding the absolute value of a Gaussian noise with mean and variance $\sigma^2 = 0.3$. c), d), and e) deleting significant parts of a fundamental memory. The following rows present the corresponding images retrieved by the AMs models. Furthermore, we included below each image in Fig. 4 the corresponding PSNR rate.
5 Concluding Remarks

In this paper, we introduced the class of max-\(C\) projection autoassociative fuzzy memories (PAFMs) for the storage and retrieval of fuzzy sets or vectors on the hypercube \([0, 1]^n\). Precisely, a max-\(C\) PAFM \(V\) is designed for the storage of a fundamental memory set \(A = \{a^1, \ldots, a^k\}\). Afterward, given an input vector \(x \in [0, 1]^n\), which can be a dilated version of fundamental memory \(a^\xi\), the max-\(C\) PAFM produces as output \(V(x)\) the projection of \(x\) into the set of all max-\(C\) combinations of \(a^1, \ldots, a^k\). Besides the formal definition, Theorem 1 provides an effective formula for computing the coefficients of the max-\(C\) combination and, thus, for the implementation of a max-\(C\) PAFM. Furthermore, we showed that a max-\(C\) PAFM exhibits optimal absolute storage capacity if the fuzzy conjunction \(C\) has a left identity.

In this paper, we also presented some computational experiments concerning the reconstruction of corrupted gray-scale images. The max-\(C\) PAFMs always outperformed the min-\(D\) AFMMs, KAM, and OLAM for the reconstruction of gray-scale images corrupted by dilative noise. For large intensities of noise, the new memories produced the largest PSNR rates. In particular, the max-\(C_M\) PAFM \(V_M\), which is based on the minimum fuzzy conjunction, is very robust in the presence of dilative noise.

In the future, we plan to investigate further the properties of the max-\(C\) PAFMs and perform more computational experiments. Furthermore, we intend to establish relations between the new models and others associative memories models from the literature.

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References


