On the Recall Capability of Recurrent Exponential Fuzzy Associative Memories Based on Similarity Measures

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Abstract. Recurrent exponential fuzzy associative memories (RE-FAMs) are non-distributive memory models derived from the multivalued exponential recurrent associative memory (MERAM) of Chiueh and Tsai. A RE-FAM defines recursively a sequence of fuzzy sets obtained by a weighted average of the fundamental memories. In this paper, we show that the output of a single-step RE-FAM can be made as close as desired to a certain convex combination of the fundamental memories most similar to the input. This paper also addresses the storage and recall capability of RE-FAMs. Precisely, computational experiments reveal that RE-FAMs can be effectively used for the retrieval of gray-scale images corrupted by either Gaussian noise or salt and pepper noise.

Keywords. Associative memory, recurrent neural network, fuzzy system, gray-scale image processing.

1. Introduction

Associative memories (AMs) are mathematical constructs motivated by the human brain ability to store and recall information [1, 2, 3, 4]. Such as the biological neural network, an AM should be able to retrieve a memorized information from a possibly incomplete or corrupted item. An AM designed for the storage and recall of fuzzy sets is called fuzzy associative memory (FAM) [5, 6]. Precisely, a FAM is designed for the storage of associations \((A^1, B^1), (A^2, B^2), \ldots, (A^p, B^p)\), where \(A^i\) and \(B^i\) are fuzzy sets for all \(\xi = 1, \ldots, p\). Afterward, the FAM model is expected to retrieve a certain \(B^\xi\) in response to the presentation of a partial or noisy version \(\tilde{A}^\xi\) of \(A^\xi\). Examples of FAM applications are pattern classification and recognition [7, 8], optimization, computer vision and image retrieval [9, 10, 11], prediction [12, 13], and control [14, 5].

Research on FAM models dates to the early 1990s with Kosko’s work [5]. Generally speaking, Kosko’s FAM stores an association \((A^\xi, B^\xi)\) in a matrix \(M^\xi\) using either the correlation-minimum or the correlation-product encoding scheme. In order to avoid crosstalk, Kosko proposed a FAM bank in which the output is determined by a weighted sum of the fuzzy sets produced separately by each FAM matrix. Specifically, if \(X\) is the input fuzzy set, the output of a FAM bank is \(Y = \sum_{\xi=1}^{\xi=p} w_\xi \tilde{Y}^\xi\), where \(\tilde{Y}^\xi\) is given by either the max-min or max-product composition of \(M^\xi\) by \(X\), for \(\xi = 1, \ldots, p\).

The separate storage of FAM matrices partially solves the crosstalk problem, but it consumes a lot of space. Thus, many researchers developed FAM models in which \(p\) associations are encoded in a single matrix. For instance, Chung and Lee proposed an encoding scheme based on the max-T composition, where \(T\) refers to a triangular norm [15]. Similarly, the implicating fuzzy learning proposed by Sussner and Valle determines a unique FAM matrix using the min-I composition, where \(I\) denotes a residual implication [16]. The content-association associative memory (ACAM) proposed recently by Bui et al. also encodes a set of associations \(\{(A^\xi, B^\xi) : \xi = 1, \ldots, p\}\) using a single matrix [11]. A comprehensive review on FAM models in which the associations are encoded in a single matrix can be found in [6, 10].

Besides the active research on matrix-based FAMs, there is an increasing interest on non-distributive FAM models such as the \(\Theta\)-FAMs introduced recently by Esmi et al. [5]. In general terms, a \(\Theta\)-FAM yields the union \(\cup_{\gamma \in \Gamma} B^\gamma\), where \(\Gamma \subseteq \{1, \ldots, p\}\) is the set of the indexes that maximizes a certain function of the input fuzzy set. For instance, a SM-FAM is obtained by considering the indexes that maximizes the similarity measure between \(A^\xi\) and \(X\).

The recurrent exponential fuzzy associative memories (RE-FAMs), previously called fuzzy exponential recurrent neural network (FERNN), also belong to the class of non-distributive models [17]. They have been derived from the multivalued exponential recurrent associative memories (MERAMs) of Chiueh and Tsai using concepts from fuzzy set theory [15]. Like MERAMs, RE-FAMs only implement autassOCIATIVE memories, that is, they are designed for the storage and recall of fuzzy sets \(A^1, \ldots, A^p\). Furthermore, RE-FAMs are recurrent models: They produce a sequence of fuzzy sets \(X_0, X_1, \ldots\) which presumably converges to the desired output. Indeed, we show that the output of a single-step RE-FAM can be made as close as desired to a certain convex combination of the fuzzy sets \(A^1, \ldots, A^p\) by increasing the parameter (basis) of the exponential.

The paper is organized as follows. Section 2 briefly reviews the MERAMs of Chiueh and Tsai. RE-FAMs and a theoretical result concerning their recall capability are discussed in Section 3. Computational experiments concerning the retrieval of corrupted gray-scale images are given in Section 4. The paper finishes with some concluding remarks in Section 5.

2. Multivalued Exponential Recurrent Associative Memories
In the early 1990s, Chieh and Goodman introduced the class of recurrent correlation associative memories, which includes the Hopfield network and some high storage capacity models such as the exponential correlation associative memories (ECAMs) as particular instances. Besides the very high storage capacity, ECAMs exhibit excellent error correction capabilities. On the downside, they are designed for the storage and recall of bipolar vectors.

Many applications of AMs, including the retrieval of grayscale images in the presence of noise, require the storage and recall of real-valued vectors or fuzzy sets. On the other hand, multi-valued vectors, representing the degree to which the element \( u \) belongs to the fuzzy set \( A \), are designed for the storage and recall of bipolar vectors. Therefore, AMs, including the retrieval of grayscale images in the presence of noise, require the storage and recall of bipolar vectors.

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### Definition 1 (Similarity Measure)

A similarity measure is a function \( \mathcal{S} : \mathcal{F}(U) \times \mathcal{F}(U) \rightarrow [0,1] \) which satisfies the following properties for any fuzzy sets \( A, B, C, D \in \mathcal{F}(U) \):

1. \( \mathcal{S}(A, B) = S(B, A) \).
2. \( \mathcal{S}(A, A) = 1 \).
3. If \( A \subseteq B \subseteq C \subseteq D \), then \( \mathcal{S}(A, D) \leq \mathcal{S}(B, C) \).
4. \( \mathcal{S}(A, \bar{A}) = 0 \), for every crisp set \( A \in \mathcal{F}(U) \).

In addition, we say that \( \mathcal{S} \) is a strong similarity measure if \( \mathcal{S}(A, B) = 1 \) implies \( A = B \).

#### Example 1

Let \( U = \{u_1, \ldots, u_n\} \) be a finite universe of discourse. The following presents three instances of strong similarity measures [19, 25, 26, 27].

1. **Gregson similarity measure**:

   \[
   \mathcal{S}_g(A, B) = \frac{\sum_{j=1}^{n} A(U_j) \wedge B(U_j)}{\sum_{j=1}^{n} A(U_j) \vee B(U_j)},
   \]

   where the symbols \( \wedge \) and \( \vee \) denote respectively the minimum and maximum operations.

2. **Eisler and Ekmken similarity measure**:

   \[
   \mathcal{S}_e(A, B) = \frac{2 \sum_{j=1}^{n} A(U_j) \wedge B(U_j)}{\sum_{j=1}^{n} A(U_j) + \sum_{j=1}^{n} B(U_j)}.
   \]

3. **Complement of the relative Hamming distance**:

   \[
   \mathcal{S}_h(A, B) = 1 - \frac{1}{n} \sum_{j=1}^{n} |A(U_j) - B(U_j)|.
   \]

A recurrent exponential fuzzy associative memory (RE-FAM) is a two-layer dynamic neural network designed for the storage of a family \( \mathcal{A} = \{A^1, A^2, \ldots, A^p\} \subseteq \mathcal{F}(U) \) of fuzzy sets [17]. The first layer computes an exponential of the similarity between \( A^t \) and the current state, represented by a fuzzy set \( X_t, x_j \in \mathcal{F}(U), \) for each \( x_j \in \{1, \ldots, p\} \). The output layer yields a convex combination of \( A^1, \ldots, A^p \) whose weights are the outputs of the previous layer. Figure 1 shows a block diagram of a RE-FAM. Formally, a RE-FAM is defined as follows:

![Figure 1. Block diagram of a RE-FAM.](image-url)
Definition 2 (RE-FAM) Consider a real number \( \alpha > 0 \), fuzzy sets \( A^1, \ldots, A^p \in \mathcal{F}(U) \), and let \( \mathcal{S} : \mathcal{F}(U) \times \mathcal{F}(U) \rightarrow [0, 1] \) denote a fuzzy similarity measure. Given a fuzzy set \( X_0 \in \mathcal{F}(U) \), a RE-FAM produces a sequence \( \{X_t\} \) of fuzzy sets according to the following evolution equation for all \( u \in U \):

\[
X_{t+1}(u) = \frac{\sum_{\xi=1}^{p} A^\xi(u)e^{\alpha \mathcal{S}(A^\xi, X_t)}}{\sum_{\eta=1}^{p} e^{\alpha \mathcal{S}(A^\eta, X_t)}}, \quad \forall t = 0, 1, \ldots
\]  

(6)

The following theorem shows that the output of a single-step RE-FAM converges point-wise to a convex combination of certain fundamental memories as the parameter \( \alpha > 0 \) tends to infinity. In other words, the fuzzy set \( X_1 \) can be made as close as desired to a certain convex combination of \( A^1, \ldots, A^p \) by choosing a sufficiently large parameter \( \alpha > 0 \).

Theorem 1 Consider a family of fuzzy sets \( \{A^1, \ldots, A^p\} \subseteq \mathcal{F}(U) \) and let \( \mathcal{S} \) denote a similarity measure. Given a fuzzy set \( X_0 \in \mathcal{F}(U) \), let \( \Gamma \subseteq \{1, \ldots, p\} \) be the set of the indexes of \( A^1, \ldots, A^p \) with the largest similarity degree with \( X_0 \), that is,

\[
\Gamma = \{ \gamma : \mathcal{S}(A^\gamma, X_0) \geq \mathcal{S}(A^\xi, X_0), \forall \xi = 1, \ldots, p \}.
\]  

(7)

If \( X_1 \in \mathcal{F}(U) \) denotes the output of a single-step RE-FAM given by (6) with \( t = 0 \), then

\[
\lim_{\alpha \to \infty} X_1(u) = \frac{1}{\text{Card}(\Gamma)} \sum_{\gamma \in \Gamma} A^\gamma(u), \quad \forall u \in U.
\]  

(8)

Proof. Let \( \upsilon = \max_{\xi=1}^{p} \{\mathcal{S}(A^\xi, X_0)\} \) denote the largest similarity degree between the input fuzzy set \( X_0 \) and \( A^1, \ldots, A^p \). From (7), we have \( \upsilon = \mathcal{S}(A^\gamma, X_0) \) for all \( \gamma \in \Gamma \) while \( \mathcal{S}(A^\xi, X_0) < \upsilon \) if \( \xi \notin \Gamma \). Now, multiplying both numerator and denominator of (6) by \( e^{-\alpha \upsilon} \) and breaking up the sums, we obtain

\[
X_1(u) = \frac{\sum_{\xi=1}^{p} A^\xi(u)e^{\alpha \mathcal{S}(A^\xi, X_0) - \upsilon}}{\sum_{\eta=1}^{p} e^{\alpha \mathcal{S}(A^\eta, X_0) - \upsilon}}
\]  

(9)

\[
= \frac{\sum_{\gamma \in \Gamma} A^\gamma(u) + \sum_{\xi \notin \Gamma} A^\xi(u)e^{\alpha \mathcal{S}(A^\xi, X_0) - \upsilon}}{\sum_{\eta=1}^{p} e^{\alpha \mathcal{S}(A^\eta, X_0) - \upsilon}}.
\]

Since \( \mathcal{S}(A^\gamma, X_0) - \upsilon < 0 \) for all \( \gamma \notin \Gamma \), the second sum in both numerator and denominator vanishes as \( \alpha \to \infty \). Hence, we obtain

\[
\lim_{\alpha \to \infty} X_1(u) = \frac{\sum_{\gamma \in \Gamma} A^\gamma(u)}{\sum_{\gamma \in \Gamma} 1}, \quad \forall u \in U,
\]  

(10)

which is exactly the identity given by (8).

From Theorem 1 if \( \mathcal{S}(A^\xi, X_0) > \mathcal{S}(A^\gamma, X_0) \) for all \( \xi \neq \gamma \), then the output of a single-step RE-FAM converges point-wise to \( A^\gamma \) as \( \alpha \to \infty \). Hence, we conjecture that the basin of attraction of \( A^\gamma \) is the region

\[
\mathcal{S}^\gamma = \{ X \in \mathcal{F}(U) : \mathcal{S}(A^\gamma, X) > \mathcal{S}(A^\xi, X), \forall \xi \neq \gamma \}.
\]  

(11)

if \( \mathcal{S} \) is a strong similarity measure, then \( A^\gamma \) is the region where \( \mathcal{S} \) is a strong similarity measure and the parameter \( \alpha \) is sufficiently large.

Remark 1 The pattern recalled by an autoassociative similarity measure FAM (SM-FAM) of Esmi et al. under presentation of \( X \in \mathcal{F}(U) \) is the fuzzy set

\[
Y = \bigcup_{\gamma \in \Gamma} A^\gamma,
\]

(12)

where \( \Gamma \) is the set of indexes given by (8). Hence, an autoassociative SM-FAM differs from a single-step RE-FAM with a large parameter \( \alpha \) in the way \( A^\gamma \) is the region

\[
A^\gamma = \{ X \in \mathcal{F}(U) : \forall \xi \neq \gamma, \mathcal{S}(A^\xi, X) < \mathcal{S}(A^\gamma, X) \}. \tag{10}
\]

4. Computational Experiments

Let us perform some experiments concerning the retrieval of corrupted gray-scale images. Consider the eight images displayed in Figure 2. Each of these images corresponds to a fuzzy set \( A^V \in \mathcal{F}(U) = \{1, \ldots, 256\} \times \{1, \ldots, 256\} \). First, we synthesized RE-FAMs designed for the storage of these eight gray-scale images by considering the parameter \( \alpha = 30 \) and the similarity measures given by (9), (4), and (5). For comparison purposes, we also stored the eight gray-scale images in the following FAM models: Lukasiewicz implicative fuzzy associative memory (IFAM) \[16\], the content-association associative memory (ACAM) with the parameter \( \eta = 2 \) \[11\], and the SM-FAM based on the 3 similarity measures given in Example 1 \[7\] \[8\]. In addition, we confronted the FAM models with the MERAMs based on \( \Psi_c \) and \( \Psi_s \) as well as the optimal linear associative memory (OLAM) \[2\], the kernel associative memory (KAM) \[28\], the complex-sigmoid Hopfield network (CSHM) \[29\], and a certain subspace projection autoassociative memory (SPAM) \[30\].
that is, when the variance or probability equals to zero. In large PSNR rates under presentation of an undistorted input, (or simulation), by the noise intensity introduced into \( \tilde{X} \) we iterated (6) until either

\[
\| A^t \|_2 > 2 \xi \mu
\]

ties varying from 0 to 0.8. In the computational experiments, scale images corrupted by Gaussian noise with zero mean observed that the IFAM and ACAM models produced whiter images as output. The MERAM based on \( \Psi_C \) retrieved the original images. Observe that the IFAM and ACAM models showed in this paper that, when \( \alpha \rightarrow \infty \), the output of a single-step RE-FAM converges to a linear combination of the fuzzy sets that are the most similar to the input. In this paper, a RE-FAM is described by a two-layer recurrent fuzzy neural network. The nodes in the first layer compute an exponential of the fuzzy similarity measure between the current state and the stored fuzzy sets. The next fuzzy set is obtained by a weighted average of the stored fuzzy sets. We showed in this paper that, when \( \alpha \rightarrow \infty \), the output of a single-step RE-FAM converges to a linear combination of the fuzzy sets that are the most similar to the input. In this paper, we also observed that the RE-FAMs may produce excellent results for the retrieval of gray-scale images corrupted by either Gaussian noise or salt and pepper noise. In particular, they outperformed the multivalued exponential recurrent associative memory (MERAM) based on \( \Psi_C \) and \( \Psi_E \) given by (2). Also, the largest PSNR rates have been obtained by the similarity measure fuzzy associative memory (SM-FAM) models of Esmai et al. Notwithstanding, by increasing the value of the parameter \( \alpha \) of the RE-FAM, we can obtain results similar to the those produced by the SM-FAMs models.

In the future, we plan investigate further the convergence of the sequence produced by an RE-FAM. We also intend to study the effect of the fuzzy similarity and other measures on the storage capacity as well as the noise tolerance of RE-FAMs. Finally, the relationship between the RE-FAMs and other fuzzy AM models, including the SM-FAMs and the KAM model, require further attention.

5. Concluding Remarks

In this paper, we investigated the fuzzy exponential recurrent neural networks (RE-FAMs), which can be used for the storage and recall of fuzzy sets. In contrast to many FAM models, a RE-FAM is described by a two-layer recurrent fuzzy neural network. The nodes in the first layer compute an exponential of the fuzzy similarity measure between the current state and the stored fuzzy sets. The next fuzzy set is obtained by a weighted average of the stored fuzzy sets. We showed in this paper that, when \( \alpha \rightarrow \infty \), the output of a single-step RE-FAM converges to a linear combination of the fuzzy sets that are the most similar to the input.

In this paper, we also observed that the RE-FAMs may produce excellent results for the retrieval of gray-scale images corrupted by either Gaussian noise or salt and pepper noise. In particular, they outperformed the multivalued exponential recurrent associative memory (MERAM) based on \( \Psi_C \) and \( \Psi_E \) given by (2). Also, the largest PSNR rates have been obtained by the similarity measure fuzzy associative memory (SM-FAM) models of Esmai et al. Notwithstanding, by increasing the value of the parameter \( \alpha \) of the RE-FAM, we can obtain results similar to the those produced by the SM-FAMs models.

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Figure 5. Input image corrupted by Gaussian noise with variance $\sigma^2 = 0.08$ followed by the images retrieved by the AM models.

Figure 6. Input image corrupted by salt and pepper noise with probability $p = 0.1$ followed by the images retrieved by the memory models.
References


